JOURNAL OF NANO- AND ELECTRONIC PHYSICS
Vol. 15 No 2, 02023(5pp) (2023)

ЖУРНАЛ НАНО- ТА ЕЛЕКТРОННОЇ ФІЗИКИ
Том 15 № 2, 02023(5cc) (2023)

New Fractional Wavelet with Compact Support and Its Application to Signal Denoising

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(Received 20 February 2023; revised manuscript received 20 April 2023; published online 27 April 2023)

Since their appearance, the wavelets have been developed very rapidly and have attracted the attention of many researchers, which resulted in the birth of several wavelet families: real, complex and fractional. However, the choice of an adequate analyzing wavelet remains an important problem; there is no wavelet suitable for all cases, for some applications, it is possible that we do not find among the known wavelets the one that suits. Therefore, it is necessary to try to build new wavelets that can adapt and cover a wide panorama of problems. In this context, we propose in the present research work a new wavelet family based on fractional calculus. The construction generally begins with the choice of an orthogonal digital low-pass filter associated with a base of wavelet with compact support; the filter will be generalized through the fractional delay (FD) Z^{-D} which is approximated by a RIF filter using the Lagrange interpolation method, while ensuring correct properties of orthogonality, compact support and regularity. Then, the high-pass fractional filter is deduced from the low-pass filter by a simple modulation. However, the scale and wavelet functions are built using the cascade Daubechies algorithm. In order to illustrate the potential and efficiency of the fractional wavelets designed within this paper compared to the different wavelets existing in the literature, an application example is presented; this is the denoising of signals by thresholding fractional wavelet coefficients. The experimental results obtained are satisfactory and promising; they show that the performance of fractional wavelets is superior to those of classical wavelets; this is due to the flexibility and high selectivity of fractional filters associated with these fractional bases.

Keywords: Fractional filters, Fractional delay, Compact support, Fractional wavelets, Signal denoising.

DOI: 10.21272/jnep.15(2).02023 PACS numbers: 43.60.Hj, 07.50.Qx

1. INTRODUCTION

In recent decades, wavelet theory has been the subject of much research, where its emergence has brought new life to all fields: mathematics, computer science, physics, geology, microelectronics, etc. In general, all the sciences that need signal analysis. Indeed, they allow covering all the disadvantages and inadequacies of the Fourier analysis, this leads them to be the ideal solution for all problems. This excessive optimism naturally led to some disappointments. In fact, the wavelet transform requires a good choice of analyzing mother wavelet, unfortunately there is no wavelet suitable for all cases, for some applications, it is possible that we do not find among the known wavelets the one that suits. In such cases, it is necessary to try to build a new wavelet that can adapt and cover a wide range of problems.

In this context, many works have been introduced with the aim of building wavelets with more elegant and efficient mathematical properties. Among the wavelets which are currently of considerable interest those based on fractional calculus. The main advantage of having a fractional order is flexibility, which allows adjustments in transform parameters such as regularity and localization of the base functions.

The idea of fractional wavelets was first proposed by Mendlovic [1] in the context of optical signal processing, where they were defined as a cascade of the classical wavelet transform and the fractional Fourier transform (FRFT) [2]. Shortly thereafter, Huang and Suter [3] proposed a fractional version of the wave packet transform (FRWPT) based on combining FRFT and the wave packet transform (WPT). Due to a lack of physical interpretation and the complexity of the calculation, the FRWPT unfortunately did not receive much attention. However, several years later, the idea was extended in the context of spline wavelets to non-integer degrees [4]. Quincunx wavelets have also been generalized to non-integer orders with a construction based on fractional Quincunx filters, which are generated through the diamond McClellan transform [5].

However, in the discrete case, the definition of the discrete fractional wavelet transform is not yet reported in the literature, therefore, a definition is consolidated by discretizing its continuous version. Recently, several fractional wavelet models have been proposed and developed, which the design differs according to the method of construction and implementation, what results a wide panorama of applications [6-12].

The aim of this paper is to introduce a new family of wavelets with compact support based on fractional calculus. The construction generally begins with the choice of an orthogonal digital low-pass filter associated with a wavelet base with compact support and then it is generalized through the fractional delay (FD) Z^{-D} [13] while ensuring correct properties of orthogonality, compact support and regularity. The high-pass filter can be built from the low-pass filter by a simple modulation, however, the associated scale and wavelet functions are deduced by the cascade algorithm [14].

The rest of this paper is organized as follows. Section 2 briefly introduces the construction principle of fractional wavelets via fractional digital delay. Section 3 presents the construction of new fractional wavelets with compact support. Section 4 examines and simu-

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lates an experimental application of the new wavelets. Finally, section 5 presents concluding remarks.

2. CONSTRUCTION PRINCIPLE OF FRACTIONAL WAVELETS VIA FRACTIONAL DIGITAL DELAY

From a structural point of view, the Discret Wavelet Transform (DWT) is identical to an iterated filter bank, which gives it a multi-resolution character, so we can use a fractional filter to implement a fractional discrete wavelet as shown in Fig. 1, where h_f,g_f, $\bar{h}_f$ and $\bar{g}_f$ are respectively fractional order transfer functions of the analysis and synthesis filters, low pass and high pass. Indeed, based on fractional operators, we can build fractional filters which lead to wavelets with interesting properties (precision, flexibility, regularity, etc.).

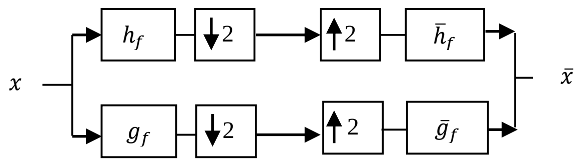


Fig. 1 – One-dimensional fractional wavelet decomposition and reconstruction

The construction generally begins with the choice of an orthogonal digital low-pass filter associated with a wavelet base with compact support; then it is generalized through the fractional delay (FD) Z^{-D}, where the delay Z^{-n} will be replaced by a fractional delay Z^{-D}, $D \in R$, while ensuring correct properties of orthogonality, compact support and regularity..

Furthermore, because of its irrational function representation, the fractional digital delay will be simulated by a filter called *the fractional delay filter*. It is then possible to construct by a simple modulation, the fractional filter passes high from the fractional filter passes low, the associated scale and wavelet functions are deduced by the cascade algorithm.

The perfect reconstruction is ensured by fractional synthesis filters, built according to the conditions of the equation [15]:

$$\begin{cases} H(w+\pi)\bar{H}(w) + G(w+\pi)\bar{G}(w) = 0 \\ H(w)\bar{H}(w) + G(w)\bar{G}(w) = 2 \end{cases} \quad (1)$$

Where H, G, $\bar{H}$ and $\bar{G}$ are respectively the transfer functions of the analysis and synthesis filters, low pass and high pass.

3. CONSTRUCTION OF FRACTIONAL WAVELETS WITH COMPACT SUPPORT

3.1 Fractional Filters Conception

The construction is based on the filters associated with the compact support wavelet bases. We will take as a starting point the case of the *Daubechies* wavelets *dbN*, where $N = 2$, i.e. *db*2.

The low-pass filter transfer function associated with the *db*2 wavelet base is given by [15]:

$$H = 0{,}483 + 0{,}836 \cdot Z^{-1} + 0{,}224 \cdot Z^{-2} - 0{,}129 \cdot Z^{-3} \quad (2)$$

The generalization of this filter to a fractional order is ensured via fractional delays Z^{-D1}, Z^{-D2} and Z^{-D3} where D1, D2 and D3 belong to the real numbers.

$$H_f = 0{,}483 + 0{,}836 \cdot Z^{-D1} + 0{,}224 \cdot Z^{-D2} - 0{,}129 \cdot Z^{-D3} \quad (3)$$

This function can be written more generally as follows:

$$H_f = \alpha + \beta \cdot Z^{-D1} + \gamma \cdot Z^{-D2} + \delta \cdot Z^{-D3} \quad (4)$$

Where *D*1, *D*2 and *D*3 represent fractional delay lengths, however α, β, γ and δ are the filter coefficients. The careful selection of these coefficients ensures the orthogonality and regularity of the scale and wavelet functions.

To simplify the calculations, we can represent the lengths of the fractional delays as follows:

$$\begin{aligned} D_1 &= 1 + m_1 \\ D_2 &= 2 + m_2, \\ D_3 &= 3 + m_3 \end{aligned} \quad (5)$$

where m_1, m_2 and m_3 represent real numbers between 0 and 2. $m_1, m_2, m_3 \in [0,2[$.

The transfer function of the filter H_f described by equation (4) will be represented as follows:

$$H_f = \alpha + \beta \cdot Z^{-1} \cdot Z^{-m_1} + \gamma \cdot Z^{-2} \cdot Z^{-m_2} + \delta \cdot Z^{-3} \cdot Z^{-m_3} \quad (6)$$

It should be noted that the filter described by the transfer function (6) not only represents the generalization of the low-pass filter associated with the *db*2 base, but it represents a generalization of all the pass- low filters associated with compact support wavelet bases that have 2 null moments such as *db*2, *coiflet*2, *symlet*2, etc.

On the other hand, it should be noted that the fractional numerical delay Z^{-m} is represented by an irrational function in which the impulse response is infinitely long. This is similar to a non- causal filter for any finite time lag, making its implementation an unfeasible task and implies a very limited implementation of the fractional filter transfer function H_f. Therefore, it is necessary to use approximation techniques to obtain a realizable implementation of the FD (Fractional Delay) and, obviously, a rational representation for the transfer function of the fractional filter H_f.

In order to design filters approximating the FD, we use the Lagrange interpolation method [13], (the choice of this method was made due to its easy implementation and good approximation). The transfer function of the FIR filter approximating the FD Z^{-m} is given by:

$$Z^{-m} \approx \sum_{n=0}^{N} h[n] Z^{-n}, \quad (7)$$

where N is the order of the filter and $h[n]$, $n = 0,1,...N$, are the real coefficients which form the impulse response of the FIR filter.

The FDF coefficients are given by:

$$h(n) = \prod_{\substack{k=0 \\ k \neq n}}^{N} \frac{m-k}{n-k} \quad pour\ n = 0,1\ldots, N \quad (8)$$

where N is the order of the filter and m is the desired delay.

According the approximation of the FD Z^{-m} by an FIR filter, the transfer function of the fractional filter H_f (equation 6) is represented by the following function:

$$H_f = \alpha + \beta \cdot Z^{-1} \cdot \left(\sum_{n=0}^{N} h[n]Z^{-n}\right) + \gamma \cdot Z^{-2} \cdot \left(\sum_{n=0}^{N} b[n]Z^{-n}\right) + \delta \cdot Z^{-3} \cdot \left(\sum_{n=0}^{N} l[n]Z^{-n}\right) \quad (9)$$

Where $h[n]$, $b[n]$ and $l[n]$ respectively represent the RIF filter coefficients for fractional delays Z^{-m1}, Z^{-m2} and Z^{-m3}, however N represents their orders. In the present paper, in order to ensure a considerable approximation precision, we choose the order of filter approximating the fractional delay $N = 2$.

The transfer function of the fractional filter H_f (equation 9) is represented by the following function:

$$H_f = \alpha + \beta \cdot Z^{-1} \cdot (h[0] + h[1] \cdot Z^{-1} + h[2] \cdot Z^{-2}) + \gamma \cdot Z^{-2} \cdot (b[0] + b[1] \cdot Z^{-1} + b[2] \cdot Z^{-2}) + \delta \cdot Z^{-3} \cdot (l[0] + l[1] \cdot Z^{-1} + l[2] \cdot Z^{-2}) \quad (10)$$

In order to ensure the regularity and orthogonality of the functions of scale and wavelet, the coefficients α, β, γ and δ are calculated according to the conditions [16-18]:

$$\forall \omega \in \mathbb{R}, \left|H_f(\omega)\right|^2 + \left|H_f(\omega + \pi)\right|^2 = 2 \quad (11)$$

$$H_f(0) = \sqrt{2} \quad (12)$$

$$\left|H_f(\pi) = 0\right| \quad (13)$$

Condition (11) is a constraint that corresponds to the orthonormality of the scale function, which also means that the H_f filter is a conjugated mirror filter [18]. The condition (12) is a simple normalization, however the condition (13) ensures the regularity of the scale and wavelet functions.

There exist a few solutions that ensure the above three conditions. In this paper, we calculated the coefficients α, β, γ and δ while ensuring the two conditions (12) and (13), however the condition (11) is ensured approximately:

$$\left|H_f(\omega)\right|^2 + \left|H_f(\omega + \pi)\right|^2 \approx 2 \quad (14)$$

$$\left|H_f(\omega)\right|^2 + \left|H_f(\omega + \pi)\right|^2 - 2 = \varepsilon \quad (15)$$

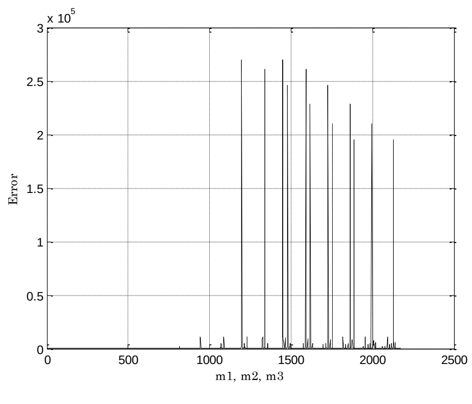


Fig. 2 – Development of the error ε in relation to $m1$, $m2$ and $m3$

The error ε plays a very important role in the choice of delays $m1$, $m2$ and $m3$. The Figure 2 shows the development of the error ε in relation to parameters $m1, m2$ and $m3$, where It appears that the optimum values of the delays $m1, m2$ and $m3$ with the smallest values of ε are concentrated in the following intervals: $m1 \in [0.1, 0.7]$, $m2 \in [0.1, 1]$, $m3 \in [0.2, 1.7]$.

The coefficients α, β, γ and δ are calculated by an algorithm based on the two conditions (12) and (13), they are linked directly to the parameters $m1, m2$ and $m3$.

$$H_f(0) = \alpha + \beta + \gamma + \delta = \sqrt{2} \quad (16)$$

$$\left|H_f(\pi)\right| = \alpha + \beta \cdot (-h[0] + h[1] - h[2]) + \gamma \cdot (b[0] - b[1] + b[2]) + \delta \cdot (-l[0] + l[1] - l[2]) = 0 \quad (17)$$

Algorithm 1

Calculation of fractional filters coefficients

Inputs: Values of fractional delay lengths $m1$, $m2$ and $m3$
Output: Fractional filters coefficients
Step 1: Begin
Step 2: Calculate the coefficients of the Lagrange interpolating filters using equation (8)
Step 3: For each value of $m1$, $m2$ and $m3$
Construct four equations system, based on equations (16) and (17)
Solve equations systems
Step 4: End For
Step 5: End

On the other hand, the high-pass fractional filter is deduced from the low-pass filter, where its coefficients are calculated by the following expression [18]:

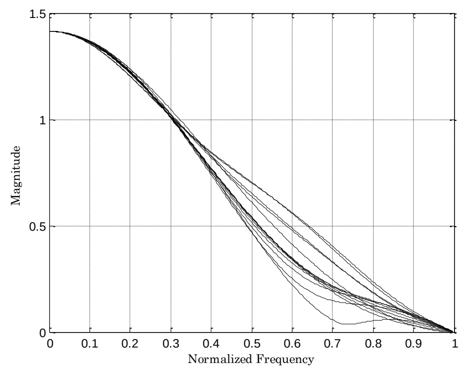
$$g_f[n] = (-1)^{N-n} h_f[N-n] \quad (18)$$

However, the synthesis fractional filters (Low-pass and High-pass) are constructed according to equation 1, and their coefficients are defined as follows:

$$\bar{h}_f[n] = h_f[N-n] \quad (19)$$

$$\bar{g}_f[n] = g_f[N-n] \quad (20)$$

The key parameters of the analysis and synthesis filters are adjusted continuously, which gives them more flexibility, more precision, thus better selectivity. All this is illustrated in Figure 3, which shows the great flexibility of the frequency responses of filters.



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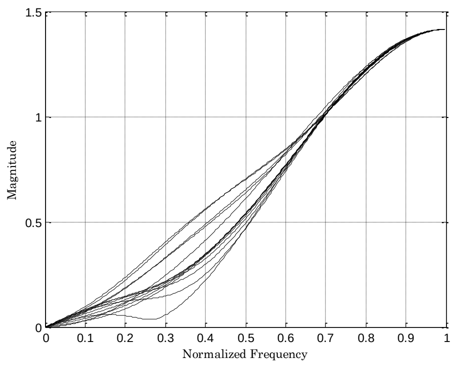

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Fig. 3 – Frequency responses of analysis and synthesis fractional filters with compact support for different values of $m1$, $m2$ and $m3$: low-pass analysis filter (a), high-pass analysis filter (b)

3.2 Scale and Wavelet Functions Construction

The scale and wavelet functions are built by the cascade algorithm [15]. They are respectively represented, according to the low-pass filters $h_f[n]$ and the high-pass filter $g_f[n]$, by the following expressions:

$$\varphi_f(x) = \sum_n h_f[n]\sqrt{2}\,\varphi_f(2x-n) \qquad (21)$$

$$\psi_f(x) = \sum_n g_f[n]\sqrt{2}\,\varphi_f(2x-n) \qquad (22)$$

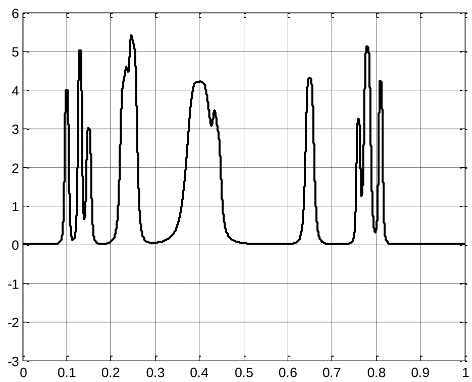
4. APPLICATION (SIGNAL DENOISING)

In order to illustrate the potential and efficiency of the proposed fractional wavelets compared to the various existing wavelets in the literature, we apply them in a signals denoising algorithm.

Denoising algorithms by thresholding wavelet coefficients are based on an estimate of the noise variance. This estimate is obtained according to a statistical calculation performed on the wavelet coefficients at the first scale. The calculated threshold from this estimate is then applied to each scale to denoise the signal.

The chosen tests signals are those constructed and analyzed in [19]: Bumps and Heavy-sine (Figure 4), these are non-stationary signals such as medical signals (ECG, EEG, EMG, etc.) seismic signals and vibration signals from rotating machines, etc.

As part of a dyadic analysis, a SureShrink type threshold (Stein's unbiased risk estimator) and a hard thresholding [20], we applied a denoising algorithm



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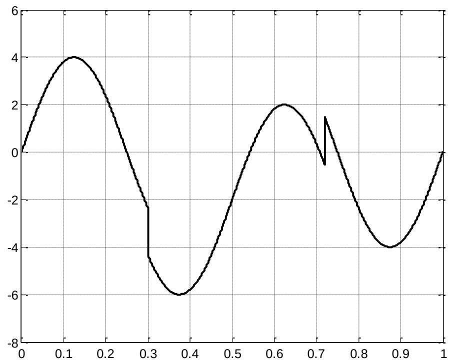

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Fig. 4 – Test signals: (a) Signal 'bumps'; (b) Signal 'Heavy sines'

based on proposed fractional wavelets on a set of noisy tests signals corrupted by Gaussian white noise of standard deviation 10, 20 and 30.

We quantify the quality of the denoising by calculating the signal-to-noise-ratio (SNR) which is defined as follows:

$$SNR = 10 \cdot log_{10} \frac{P_{signal}}{P_{noise}}, \qquad (23)$$

where P_{signal}, P_{noise} represent respectively signal and noise powers.

We compare the denoising results obtained by using the fractional wavelet developed in this paper with the results obtained by applying the most famous classical wavelets such as: Haar, Daubechies, Coiflettes, Symlettes, Meyer and bi-orthogonal wavelets, by taking into account the same characteristics, in terms of thresholding, type of threshold and level of decomposition, for all wavelets whether fractional or classical. Table 1 summarizes the best results obtained for each type of wavelet. We can see that fractional wavelets produce the best results, compared to other types of wavelets especially when it comes to an intense noise; it comes back to the great flexibility of the filters that constitute them. This important property gives to the fractional wavelets a good selectivity which implies considerable precision especially when decomposing the signals in approximations and details.

5. CONCLUSION

In this paper, a new wavelet family based on fractional delay was introduced. After the description of the principle of the construction of these new bases, we have presented the different steps of designing fractional wavelets with a compact support, where the construction is based, in general, on the generalization of a known low-pass filter whose key parameters are adjusted via the length of fractional delay.

We have also seen that these wavelets have interesting characteristics and properties, where the flexibility of the associated filters and its high selectivity offer these new bases a great ability to support several digital signals processing in a very efficient way.

In order to illustrate the potential and efficiency of the fractional wavelets designed, an application example is presented. This is the denoising of signals by thresholding fractional wavelet coefficients. The experimental results have shown that the performance of fractional wavelets is superior to that of classical wavelets including Haar wavelets, Daubechies, Coiflettes, Symlettes, Meyer and bi-orthogonal wavelets. This preference is due to the flexibility, selectivity and high precision of the filters which constitute them.

Table 1 – Global comparison of signals denoising results: Bumps and heavy-sine, using multiple wavelet families.

Input Signals	σ/SNR Noised Signal	SNR of Denoised Signal							
		Haar	*db*4	*coif*3	*sym*4	*dmey*	*bior*2.2	*rbio*1.1	Fract. with compact support
Bumps	$\sigma = 10$ (25.10)	30.40	30.70	30.78	30.36	30.35	30.72	30.59	32.83
	$\sigma = 20$ (19.09)	25.20	25.29	25.50	25.00	24.47	25.48	24.78	27.38
	$\sigma = 30$ (15.57)	20.81	21.58	21.72	20.82	21.43	22.19	21.46	24.10
Heavy-sine	σ/SNR Noised Signal	Haar	*db*2	*coif*2	*sym*3	*dmey*	*bior*2.2	*rbio*1.1	Fract. with compact support
	$\sigma = 10$ (29.79)	34.89	34.97	34.90	34.88	34.73	34.86	34.49	39.23
	$\sigma = 20$ (23.75)	28.06	27.89	28.32	28.00	27.48	28.20	28.12	32.09
	$\sigma = 30$ (20.23)	23.99	23.93	23.81	24.21	23.25	24.14	23.46	27.57

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