

About One Improvement of Approximating Functions Method for the Problems with Nonlinear Media

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Volterra integral equation method, based on integral equations equivalent to the Maxwell's equations, is an alternative to the differential formulation of the problem for modeling a wide range of electrodynamic problems. The approximating functions method, a particular case of the finite element method, plays the role of analytical-numerical component of Volterra integral equation method. It based on partitioning the definition region of the problem by cells and on the approximation of the desired solution by orthogonal polynomials. This process leads to constructing a system of nonlinear algebraic equations, which is the result of calculating the original Volterra integral equation at the mesh points. Its computational efficiency can be significantly improved by dividing each equation of the system into a set of blocks that can be calculated in advance.

This article presents a modification of the approximating functions method for solving problems of electrodynamic in one spatial dimension and time domain using the approach of Volterra integral equations. The main purpose of the modification is to increase the speed of computations and reduce the consumption of computer resources, which is especially important when considering problems with nonlinear media. It is proved that the proposed modification does not violate the algorithm of the method and does not lead to an increase in the error. The proposed method is applied to the problems of interaction of electromagnetic pulses of three different types: simple Gaussian pulse, single cycle Gaussian pulse and oscillated Gaussian pulse – incident on a layer with a second-order nonlinear medium, placed in a linear environment. The obtained simulation results are analyzed, estimates of the reduction in computation time and errors are presented.

Keywords: Nonlinear dielectric layer, Approximating functions method, Volterra integral equation method, Computational efficiency.

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1. INTRODUCTION

The study of the phenomena that appear during the interaction of electromagnetic fields with planar-layered structures with nonlinear media is of great importance, since this is a fundamental process in signal conversion technologies that are used, for example, in optical communication, nanocomputers and other modern applications of electrodynamic, where electromagnetic waves interact in certain time with the media in bounded spatial regions. The study and modeling of such initial-boundary problems requires the development of adequate mathematical models and the construction of appropriate analytical, numerical or hybrid analytical-numerical methods.

The Volterra integral equation method is an approach based on integral equations equivalent to the original Maxwell's equations, with key features [1, 2]: a natural description of non-stationary and nonlinear properties of media inside and outside the inhomogeneity in the environment, a unified definition of problems inside and outside it, as well as the inclusion of the initial and boundary conditions in the same equation. Together, they simplify problem formulation and solution.

Application of this method to non-stationary problems leads to the construction of the Volterra integral equation of the second kind [3, 4], an approximate solution for which can be obtained using the approximating functions method proposed in publications [5-7], which is a further development of works [8, 9].

The approximating functions method itself is a special case of the finite element method [10, 11] in the spatial and temporal domain [12] and therefore is based on dividing the domain of the problem by a mesh into cells and approximating the desired function by four types of Lagrange polynomials of the second order in each of them. Calculation of the resulting equation at all points of the mesh reduces the solution of the problem inside the inhomogeneity to finding the solution of a system of nonlinear algebraic equations, which is solved by Newton's method [13].

Such generation of a separate equation for each mesh point is very costly [14]. And with an increase in the nonlinearity of the inhomogeneity, their influence increase respectively. This article shows that the parts of the equations that are used in this process and are computationally intensive are common. Dividing each equation into a set of blocks that can be calculated in advance can reduce the consumption of computer resources and increase the computational performance of the process of constructing a system's equations. Identifying these blocks and developing a way to calculate them efficiently is the goal of this article.

2. FORMULATION OF THE PROBLEM

According to the Volterra integral equation method [5-6], the problem of interaction of an electromagnetic wave with a dielectric layer in the space-time domain is formulated as an integral Volterra equation of the second kind:

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$$E(t, x) = E_0(t, x) - \frac{1}{2v} \frac{\partial}{\partial t} \int_0^\infty dt' \int_0^L dx' \chi(x') \delta\left(t - t' - \frac{|x - x'|}{v}\right) \times \left[\frac{1}{\varepsilon \varepsilon_0} P(t', x') - \frac{\varepsilon - 1}{\varepsilon} E(t', x') \right]. \quad (1)$$

Equation (1) fully describes the evolution of the electromagnetic field in the space-time domain $(t, x) \in [0, \infty) \times [0, L]$. The unperturbed initial field outside the inhomogeneity $[0, L]$ is described by $E_0(t, x)$, the unknown field inside it is described by $E(t, x)$, and $P(t, x)$ is a polarization of medium of the inhomogeneity. ε is the dielectric constant of the environment, $v = c/\sqrt{\varepsilon}$ and c is the speed of light in a vacuum. Also, Equation (1) uses the SI system of units.

Integration over the spatial coordinate using δ (Dirac delta function) with the introduction of dimensionless variables $(\tau, \xi) \in [0, \infty) \times [0, 1]$ gives the following equation:

$$E(\tau, \xi) = E_0(\tau, \xi) - \frac{1}{2} \frac{\partial}{\partial \tau} \left[\int_{\tau_{\min}^L}^{\tau} \left(\frac{1}{\varepsilon \varepsilon_0} P(\tau', \xi - \tau + \tau') - \frac{\varepsilon - 1}{\varepsilon} E(\tau', \xi - \tau + \tau') \right) d\tau' + \int_{\tau_{\min}^H}^{\tau} \left(\frac{1}{\varepsilon \varepsilon_0} P(\tau', \xi + \tau - \tau') - \frac{\varepsilon - 1}{\varepsilon} E(\tau', \xi + \tau - \tau') \right) d\tau' \right], \quad (2)$$

where $\tau = vt/L$, $\xi = x/L$ and $\tau_{\min}^L = \max(0, \tau - \xi)$, $\tau_{\min}^H = \max(0, \tau + \xi - 1)$.

Equation (2) has nonlinearity, which is introduced by the expression for the polarization P , which describes the nonlinear features of the medium in the inhomogeneity. In dimensionless variables polarization has a form:

$$P(\tau, \xi) = \varepsilon_0 (\varepsilon_1 - 1) E(\tau, \xi) + \sum_{i=2}^n \gamma_i E^i(\tau, \xi), \quad (3)$$

where ε_1 is the permittivity and γ_i are nonlinear susceptibilities of the i -th order of the inhomogeneity.

3. INTRODUCTION OF GENERAL EXPRESSIONS

To continue let us introduce the functions for the first and the second part in Equation (2):

$$I_L(\tau, \xi, F) = \frac{\partial}{\partial \tau} \int_{\tau_{\min}^L}^{\tau} F(\tau', \xi - \tau + \tau') d\tau', \quad (4)$$

$$I_H(\tau, \xi, F) = \frac{\partial}{\partial \tau} \int_{\tau_{\min}^H}^{\tau} F(\tau', \xi + \tau - \tau') d\tau', \quad (5)$$

where

$$F(\tau, \xi) = \sum_{i=1}^n \tilde{\gamma}_i E^i(\tau, \xi) \quad (6)$$

with $\tilde{\gamma}_1 = (\varepsilon_1 - \varepsilon)/\varepsilon$ and $\tilde{\gamma}_i = \gamma_i/(\varepsilon \varepsilon_0)$ for $i \geq 2$.

In this notations Equation (2) will be written in the next short form:

$$E(\tau, \xi) = E_0(\tau, \xi) - \frac{1}{2} [I_L(\tau, \xi, F) + I_H(\tau, \xi, F)]. \quad (7)$$

4. THE APPROXIMATING FUNCTIONS METHOD MODIFICATION

According to approximating functions method [5-6], the domain of problem statement D of Equation (7) should be divided into a mesh of semi-closed squares with side h :

$$D = \cup D_{ij}, \quad D_{ij} = \{ih \leq \tau < (i+1)h, jh \leq \xi < (j+1)h\}, \quad (8) \quad i = \overline{0, n-1}, j = \overline{0, m-1}.$$

An approximate solution to Equation (7) in each square D_{ij} is constructed as a piecewise smooth function $\hat{E}_{i,j}(\tau, \xi)$, which is constructed from four interpolation polynomials multiplied by the corresponding coefficients $c_{i,j}$:

$$\hat{E}_{i,j}(\tau, \xi) = \sum_{d_1=0}^1 \sum_{d_2=0}^1 c_{i+d_1, j+d_2} \cdot T_{i,j}^{d_1, d_2}(\tau, \xi). \quad (9)$$

Interpolation polynomials $T(\cdot)$ are in the form of Lagrange polynomials of the second order and are continuous with their first derivatives on the cells border [5-6], and in general can be written as follows:

$$T_{i,j}^{d_1, d_2}(\tau, \xi) = T_i^{d_1}(\tau) \cdot T_j^{d_2}(\xi), \quad (10)$$

$$T_i^d(s) = (1-d) + (-1)^{d+1} \frac{s-ih}{h}, d = \overline{0, 1}. \quad (11)$$

Obviously, the advantage of these polynomials is that they represent an explicit form of the product of simple polynomials, each of which depends on only one variable, which makes them relatively easy to use in analytical operations, including integration and differentiation.

The complete approximate solution to Equation (7) has the form of the sum of Equation (9) for all cells in the mesh D_{ij} :

$$E(\tau, \xi) \approx \hat{E}(\tau, \xi) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \hat{E}_{i,j}(\tau, \xi) \quad (12)$$

Substitution of Equation (12) and (9) in Equation (7) and calculation of the resulting function at all mesh vertexes (τ_i, ξ_j) , $i = \overline{0, n}, j = \overline{0, m}$ gives the system of algebraic equations $\{\Psi_{i,j}\}$ for the unknown coefficients

$c_{i,j}$, which is solved by Newton method.

Let's write the resulting system of nonlinear equations in the form of Equation (7):

$$\Psi_{i,j}: \quad c_{i,j} - E_0(\tau_i, \xi_j) + \frac{1}{2} [\hat{I}_L(\tau_i, \xi_j) + \hat{I}_H(\tau_i, \xi_j)] = 0, \quad (13)$$

where $\hat{I}_{L,H} = I_{L,H}(\tau, \xi, \hat{F})$, $i = \overline{0, n}$, $j = \overline{0, m}$, \hat{F} is the Equation (6) that depends on $\hat{E}(\tau, \xi)$.

5. SEPARATION COMMON BLOCKS IN EXPRESSIONS

Let's rewrite the minimal time values from absolute coordinates to index ones:

$$i_{\min}^L = \left\lfloor \frac{\tau_{\min}^L}{h} \right\rfloor = \begin{cases} 0, & i \leq j \\ i - j, & i > j \end{cases}, \quad (14)$$

$$i_{\min}^H = \left\lfloor \frac{\tau_{\min}^H}{h} \right\rfloor = \begin{cases} 0, & i \leq m - j \\ i + j - m, & i > m - j \end{cases}, \quad (15)$$

where $m = \lfloor 1/h \rfloor$ is the width of inhomogeneity in index units (from Equation (8)). And then let us consider the $\hat{I}_L(\tau_i, \xi_j)$ and $\hat{I}_H(\tau_i, \xi_j)$ integrals and split their whole integration intervals $[\tau_{\min}^L, \tau_i]$ and $[\tau_{\min}^H, \tau_i]$ to the sum of segments of length h on the mesh:

$$[\tau_{\min}^L, \tau_i] = \cup [kh, (k+1)h], \quad k = \overline{i_{\min}^L, i-1}, \quad (16)$$

$$[\tau_{\min}^H, \tau_i] = \cup [kh, (k+1)h], \quad k = \overline{i_{\min}^H, i-1}. \quad (17)$$

Then each integral will be represented as a sum of integrals for each segment. The first integral in this sum has a lower limit τ_{\min} that can be equals 0 or depends on the position of the point (τ_i, ξ_j) as it shown in Equation (2), and has a constant upper limit. The last integral in the sum has a constant lower limit and τ as upper limit. All integrals in the middle have the both constant limits.

Graphically the path of integrating for $\hat{I}_L(\tau_i, \xi_j)$ and $\hat{I}_H(\tau_i, \xi_j)$ from the Equation (13) on the mesh from Equation (8) can be represented as shown on Fig. 1.

Formatting of the lines on Figure 1 means the constant or variable limits in $\hat{I}_L(\tau_i, \xi_j)$ and $\hat{I}_H(\tau_i, \xi_j)$ in the cell (i, j) : blue dot-line – both limits are constant; yellow solid line – lower limit depends on τ and ξ , upper limit is constant; red dash line – lower limit is constant, upper limit is τ ; yellow-red solid-dash line – both limits depends on τ or ξ . Separation line, where the lower limit of the first integral in the sum becomes from 0 to some function $m(\tau, \xi)$ represented as a gray

dash-dot “diagonal” line on Fig. 1: if the start point of integration lies to the left of the “diagonal” (cases “a”, “b”, “c”) the lower limit equals 0, otherwise it depends on τ and ξ (cases “d”, “e”, “f”).

In special one-cell cases (“a” and “f” lines on Fig. 1) there will be a single integral with the lower limit as in the first integral in the sum and the upper limit as in the last one. Also if interval is a point on the area border: $(\tau, 0)$ or $(0, \xi)$ – then the integral equals 0.

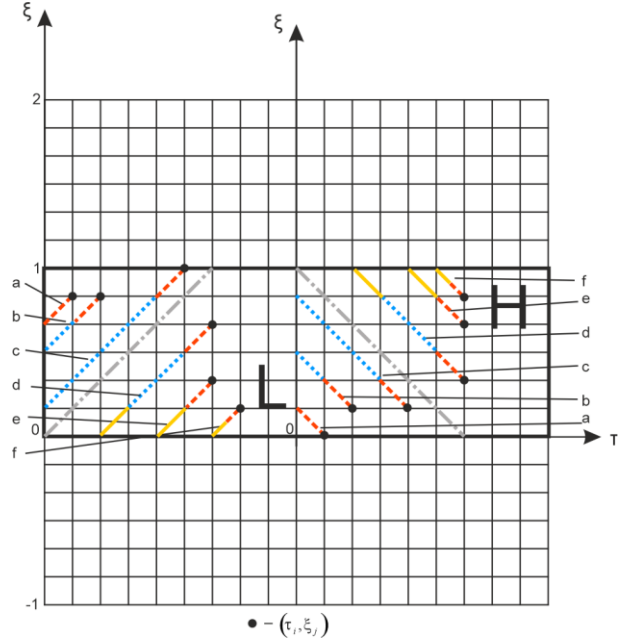


Fig. 1 – Integration paths for \hat{I}_L (left) and \hat{I}_H (right) on the same area of inhomogeneity

Let's introduce the general functions Φ as a one-cell part of $\hat{I}_L(\tau_i, \xi_j)$ and $\hat{I}_H(\tau_i, \xi_j)$:

$$\Phi_L^{[m_1, m_2]}(\tau_i, \xi_j) = \frac{\partial}{\partial \tau} \int_{m_1(\tau, \xi)}^{m_2(\tau, \xi)} \hat{F}(\tau', \xi - \tau + \tau') d\tau' \Big|_{\substack{\tau=ih \\ \xi=jh}} \quad (18)$$

$$\Phi_H^{[m_1, m_2]}(\tau_i, \xi_j) = \frac{\partial}{\partial \tau} \int_{m_1(\tau, \xi)}^{m_2(\tau, \xi)} \hat{F}(\tau', \xi + \tau - \tau') d\tau' \Big|_{\substack{\tau=ih \\ \xi=jh}} \quad (19)$$

Then $\hat{I}_L(\tau_i, \xi_j)$ for one-cell case will be written as

$$\hat{I}_L^{tc}(\tau_i, \xi_j) = \begin{cases} \Phi_L^{[0, \tau]}(\tau_i, \xi_j), & i \leq j \\ \Phi_L^{[\tau - \xi, \tau]}(\tau_i, \xi_j), & i > j \end{cases}, \quad (20)$$

For other cases it will have the form:

$$\hat{I}_L^{tc}(\tau_i, \xi_j) = \hat{I}_L^{tc}(\tau_i, \xi_j) + \sum_{k=(i_{\min}^L+1)}^{i-2} \Phi_L^{[kh, (k+1)h]}(\tau_i, \xi_j) + \Phi_L^{[(i-1)h, \tau]}(\tau_i, \xi_j), \quad (21)$$

where

$$\hat{I}_L^{tc}(\tau_i, \xi_j) = \begin{cases} \Phi_L^{[0, (i_{\min}+1)h]}(\tau_i, \xi_j), & i \leq j \\ \Phi_L^{[\tau-\xi, (i_{\min}+1)h]}(\tau_i, \xi_j), & i > j \end{cases}. \quad (22)$$

Finally the whole expression can be written in form that depends on index coordinates:

$$\hat{I}_L(\tau_i, \xi_j) = \begin{cases} 0, & i = 0, j = \overline{0, m} \\ & \text{or } i \geq 1, j = 0 \\ \hat{I}_L^{tt}(\tau_i, \xi_j), & i = 1, j = \overline{1, m} \\ & \text{or } i \geq 2, j = 1 \\ \hat{I}_L^{tct}(\tau_i, \xi_j), & i \geq 2, j = \overline{2, m} \end{cases} \quad (23)$$

For the $\hat{I}_H(\tau_i, \xi_j)$ same will be written in form

$$\hat{I}_H(\tau_i, \xi_j) = \begin{cases} 0, & i = 0, j = \overline{0, m} \\ & \text{or } i \geq 1, j = m \\ \hat{I}_H^{tt}(\tau_i, \xi_j), & i = 1, j = \overline{0, m-1} \\ & \text{or } i \geq 2, j = m-1 \\ \hat{I}_H^{tct}(\tau_i, \xi_j), & i \geq 2, j = \overline{0, m-2} \end{cases} \quad (24)$$

where

$$\hat{I}_H^{tt}(\tau_i, \xi_j) = \begin{cases} \Phi_H^{[0, \tau]}(\tau_i, \xi_j), & i \leq m-j \\ \Phi_H^{[\tau+\xi-1, \tau]}(\tau_i, \xi_j), & i > m-j \end{cases}, \quad (25)$$

$$\hat{I}_H^{tct}(\tau_i, \xi_j) = \hat{I}_H^{tc}(\tau_i, \xi_j) + \sum_{k=(i_{\min}+1)}^{i-2} \Phi_H^{[kh, (k+1)h]}(\tau_i, \xi_j) + \Phi_H^{[(i-1)h, \tau]}(\tau_i, \xi_j), \quad (26)$$

$$\hat{I}_H^{tc}(\tau_i, \xi_j) = \begin{cases} \Phi_H^{[0, (i_{\min}+1)h]}(\tau_i, \xi_j), & i \leq m-j \\ \Phi_H^{[\tau+\xi-1, (i_{\min}+1)h]}(\tau_i, \xi_j), & i > m-j \end{cases}, \quad (27)$$

The integration limits in Φ_L explicitly can have one of the next values: $m_1(\tau, \xi) \in \{0, ih, \tau - \xi\}$ and $m_2(\tau, \xi) \in \{ih, \tau\}$. The same for Φ_H will be $m_1(\tau, \xi) \in \{0, ih, \tau + \xi - 1\}$ and $m_2(\tau, \xi) \in \{ih, \tau\}$. Or in general form:

$$m_{1,2}(\tau, \xi) = a \cdot \tau + b \cdot \xi + c, \quad (28)$$

where $a \in \{0, 1\}$, $b \in \{-1, 0, 1\}$, $c \in \{-1, 0, ih\}$. In other words, the integration limits are polynomials of the first degree.

Equation (18) and (19) with simple constant or polynomial limits and polynomial integrand, which we can get explicitly by substitution Equation (9) into Equation (6), are always can be evaluated into a polynomial expression. The form of final expressions will depend on the form of the polarization defined by Equation (3).

So, for a particular form of Equation (3), Equation (18) and (19) can be explicitly calculated in general form, without substituting exact values of index coordinates

(i, j) , in advance, and will only have simple arithmetic operations, which are the fastest for calculating in computer software (for example, as shown in [15]). After storing these final expressions, the building process of Equation (23) and Equation (24) (and hence also system of Equation (13)) will be faster and less demanding on resources because of using of pre-calculated polynomial blocks.

6. APPLICATION AND VERIFICATION OF THE METHOD

The method was tested using the mathematical package Wolfram Mathematica [16] with its built-in caching capabilities for calculated expressions.

To validation, the developed approach was applied to a known problem of interaction of a plane electromagnetic wave with a dielectric layer that has an exact solution as well as the exact values of the reflection and transmission coefficients. The following parameters was used: $\varepsilon_1 = 3$, $\varepsilon = 1$, $\tau \in [0, 40]$, mesh step $h = 0.02$. The error of the obtained values of the coefficients is 0.0011%.

To verification the accuracy of the approach in the case of nonlinear problems, it was applied to the problems of passing the several Gaussian pulses through the layer with the quadratic nonlinear medium described by the Equation (3) with $\gamma_2 > 0$. The following electromagnetic characteristics was used for the medium and the layer: $\varepsilon_1 = 11$, $\varepsilon = 9$, mesh step $h = 0.005$. The correctness of the solution was checked by energy imbalance (general expressions taken from [5]), which, if the solution is correct, should tend to zero.

The evolution of simple Gaussian pulse $E(\tau, \xi) = \exp(-(\tau - \tau_0 - \xi)^2 / 2\sigma^2)$ in time interval $\tau \in [0, 15]$ and with $\gamma_2 = 1$ is shown on Fig. 2, where multiple reflections from the layer boundaries and pulse broadening due to the nonlinearity of the layer medium are visible.

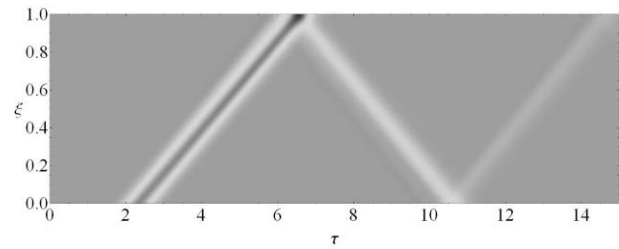


Fig. 2 – The transformation of the simple Gaussian pulse

The energy flow imbalance maximum is 6.71 % with a median of 1.63 %. This agrees well with the results from [5], and the speed of calculating the problem has increased 12 times compared to it.

The single cycle Gaussian pulse $E(\tau, \xi) = -(\tau - \tau_0 - \xi) / \sigma^2 \exp(-(\tau - \tau_0 - \xi)^2 / 2\sigma^2)$ was modeled in time interval $\tau \in [0, 10]$ with nonlinear parameter of the layer medium $\gamma_2 = 0.3$. The result is shown on Fig. 3, where presents much more beam diffusion after each reflection from the layer boundaries

and pulse form distortion due to the nonlinearity of the layer medium.

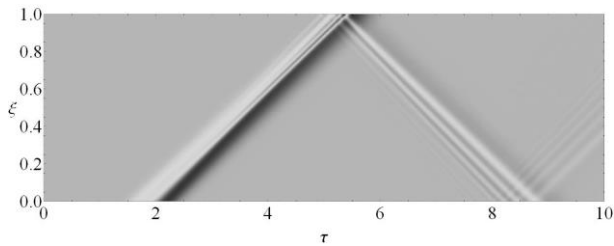


Fig. 3 – The transformation of the single cycle Gaussian pulse

After each reflection the pulse shape becomes more and more similar to the Airy pulse [17]. Also the speed of wave propagation in the layer slows down.

The energy flow imbalance maximum is 6.07 % with a median of 1.05 %.

The evolution of oscillated Gaussian pulse $E(\tau, \xi) = \cos(\eta(\tau - \tau_0 - \xi)) \exp(-(\tau - \tau_0 - \xi)^2 / 2\sigma^2)$ in time interval $\tau \in [0, 10]$ with $\gamma_2 = 1$ and normalized frequency $\eta = 10$ is shown on Fig. 4.

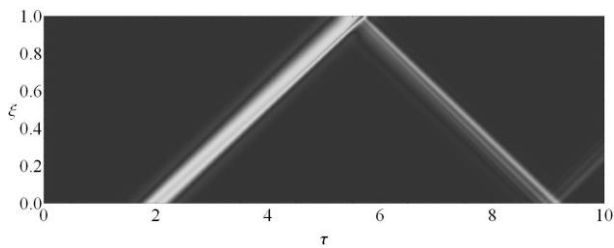


Fig. 4 – The transformation of the oscillated Gaussian pulse

The speed of wave propagation in the layer also slows down. Each reflected wave loses its incident structure and takes a "sawtooth" shape with beam diffusion. The energy flow imbalance maximum is 5.24 % with a median of 1.17 %.

7. MATERIALS AND METHODS

To implement the calculation experiments, Wolfram Mathematica [16] system was used, since this software is one of the most versatile solutions for modeling dynamic processes of various natures [15, 18].

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In addition, for algorithms such as the finite element method, it has proven to be useful [19, 20]. Also, it has built-in caching capabilities for calculated expressions [21].

The Mathematica document had its own kernel of the mathematical processor, which was allocated its own CPU core for guaranteed independent processing.

Cloud technologies of DigitalOcean were chosen as a platform for deployment, where servers based on Ubuntu 20.04 LTS x64 OS were located.

8. CONCLUSION

The approach to improving computational efficiency of approximating functions method for Volterra integral equation approach is developed for solving one-dimensional space and time electrodynamics problems with nonlinear media.

It is based on improving the process of constructing a system of nonlinear algebraic equations, which is the result of calculating the original Volterra integral equation at the mesh points, by dividing each integral in equation into a sum of sub-integrals for one mesh cell. Each of these sub-integrals can be of one of the 12 types and can be calculated in general form in advance, without substituting exact values of mesh coordinates and mesh step size. Because of the polynomial structure of the approximating functions, the expression of the polarization and integration limits the final expressions will only have simple arithmetic operations, which are the fastest for calculating in computer software. Once evaluated, these expressions can be stored not only in RAM, but also in external memory for reuse.

The approach was tested using the mathematical package Wolfram Mathematica with its built-in caching capabilities for calculated expressions.

The transformations of a several types Gaussian pulses by a layer with a second-order nonlinear features is considered. The estimation of errors shows good results. Due to the application of the modified method, the speed of calculating the known problem is 12 times increased. The actual rate of increase in speed depends on the nonlinearity of the medium in the inhomogeneity, the time interval under consideration as well as the number of parallel threads in the computation, the programming language and other characteristics of the computing system.

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Про одне вдосконалення методу апроксимуючих функцій для задач з нелінійними середовищами

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Метод інтегральних рівнянь Вольтерра, заснований на інтегральних рівняннях, еквівалентних рівнянням Максвелла, є альтернативою диференційній постановці задачі для моделювання широкого загалу задач електродинаміки. Метод апроксимуючих функцій, що є окремим випадком методу кінцевих елементів, відіграє роль аналітико-чисельної складової методу інтегральних рівнянь Вольтерра. Він заснований на розбитті області визначення задачі сіткою на осередки та апроксимації шуканого рішення ортогональними поліномами. Цей процес призводить до побудови системи нелінійних алгебраїчних рівнянь, яка є результатом обчислення вихідного інтегрального рівняння Вольтерра у вузлах сітки. Його обчислювальну ефективність можна значно підвищити, розбивши кожне рівняння системи на набір блоків, які можна обчислити заздалегідь.

У цій статті представлено модифікацію методу апроксимуючих функцій для вирішення задач електродинаміки в одновимірній просторовій та часовій області методом інтегральних рівнянь Вольтерра. Основною метою модифікації є збільшення швидкості обчислень та зниження споживання ресурсів комп'ютера, що особливо важливо під час розгляду задач із нелінійними середовищами. Доведено, що запропонована модифікація не порушує алгоритм методу і не призводить до збільшення похибки. Запропонований метод застосований до задач взаємодії електромагнітних імпульсів трьох різних типів: простий гаусівський імпульс, однотактний гаусівський імпульс та осцилюючий гаусівський імпульс, – що падають на шар з нелінійним середовищем другого порядку, поміщений у лінійне середовище. Проаналізовано отримані результати моделювання, надано оцінки скорочення часу обчислень та помилок.

Ключові слова: Нелінійний діелектричний шар, Метод апроксимуючих функцій, Метод інтегрального рівняння Вольтерра, Ефективність обчислень.