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Using Orthogonal Legendre Polynomials for Filtering Noisy Signals over a Limited Interval in Coordinate Space

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The paper shows that orthogonal Legendre polynomials in the interval [– 1, 1] can be effectively used to filter noisy signals, including filtering interferograms and phase maps in digital holographic interferometry. They can also be used to effectively approximate harmonic signals, and the approximation accuracy increases with the number of polynomials used. Filtering is based on the use of the optimal number of Legendre polynomials when approximating the signal. It is impractical to filter directly digital holograms and phase maps, since in this case it is necessary to use several hundred polynomials, which significantly increases the time of numerical calculations. Therefore, in digital holographic interferometry, it is necessary to filter directly the field amplitudes calculated from the digital hologram. Interferograms and phase maps can be calculated using filtered field amplitudes for different states of the object under study. If for the real or imaginary part of the signal the minimum distance between adjacent local minima (maxima) is equal to Δl, then for a satisfactory approximation of such a signal by Legendre polynomials, $6/\Delta l$ polynomials are required. The efficiency of filtering by Legendre polynomials is higher if the noise signal contains harmonic components with a frequency greater than the frequency of the useful signal.

Keywords: Digital holography, Digital holographic interferometry, Noise, Signal filtering, Legendre polynomials, Orthogonality of polynomials.

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1. INTRODUCTION

In many areas of technology, as well as in scientific research, the information noise contaminated signals are received, and it is sometimes impossible to obtain the necessary information with high validity using such noisy signals. They can be signals in the form of low voltage or low current at the output of the electronic circuit. It is known that elements of an electronic circuit, in particular resistors, *p-n* junctions of semiconductor elements, are a source of noise and the average power of which may exceed the power of signal.

Noise is also present in optical systems which impairs the visual perception of images. Besides, it is not always possible to make a reliable quantitative estimate in the presence of noise. Particularly intense noises occur in optical holography and holographic interferometry, because in most of cases the holographic process concerns objects with diffuse reflection [1].

Digital holography (DH) has an important place in metrology for the study of deformations, displacements, surface relief and visualization of phase inhomogeneities [2, 3]. In DH, the first stage in obtaining a hologram on a photosensitive medium is a purely optical stage. Photosensitive arrays based on charge-coupled devices (CCD) are usually used as a recording medium. The next step is to reproduce the original image of the research object with computer simulation of the interaction of the conjugate reference beam with a digital hologram. The reproduced image is displayed on the screen. It is obvious that the principles of DH are used in digital holographic interferometry (DHI), in particular, in the two-exposure method. DHI compares two reconstructed phase fields of one object which are recorded after a certain time interval. The difference in phase distributions determines a phase map containing phase jumps of π. The quantitative values of the measured values (displacement, deformation, vibration, etc. on the basis of the calculated phase pattern) are obtained.

The speckle decorrelation occurring between two digital holograms in the two-exposure method is in the process of DHI. This leads to the appearance of characteristic noise in the obtained phase pattern [4-6]. Speckle noise significantly distorts the quality of the obtained DHI, which reduces the resolution and accuracy of measurements [1, 2].

To improve the quality of interferograms, it is necessary to filter the noise, and filtering should be performed without distorting the interferograms. Currently, DHI consists of a number of developed approaches to reduce speckle noise. They can be divided into two types: optical and digital [7-11]. Moreover, several digital filtering methods are used to reduce speckle noise in DHI [12-15], in particular, wavelet filtering [13] and frequency domain filtering using Fourier transform [15].

In the articles [16, 17], the use of orthogonal Chebyshev polynomials [18] for DHI filtering in the case of the simplest deformation of a rough reflecting surface, for which the magnitude of the deformation depends on only one coordinate, was proposed. The quality of DHI and the corresponding phase maps was significantly improved by filtering with Chebyshev polynomials. However, the filtering of signals by orthogonal Chebyshev polynomials is complicated by the fact that the weight factor has a singularity at $x = \pm 1$ in the numerical integration belonging the interval [– 1, 1].

In this work, we studied the possibility of using orthogonal Legendre polynomials [18] to filter (improve the quality) of noise signals. They can be used for filtering digital interferograms of reflective rough surfaces due to deformation. The weight factor for these polynomials is 1. Therefore, the problems which arise when filtering by Chebyshev polynomials are eliminated.

2. LEGENDRE POLYNOMIALS AND THEIR PROPERTIES

Legendre polynomials can be represented as the following recurrence relations [18, 19]:

$$P_0(x)=1,\ P_1(x)=x,$$
$$P_{n+1}(x)=\frac{2n+1}{n+1}xP_n(x)-\frac{n}{n+1}P_{n-1}(x). \quad (1)$$

Polynomials with paired numbers are described by symmetric functions, and polynomials with odd numbers are described by antisymmetric functions. In addition, the polynomial number determines the number of intersections of the abscissa axis by the polynomial in the interval [– 1, 1].

These polynomials are orthogonal with weight 1 in the interval [– 1, 1], and the conditions of orthogonality for them are the following [18-20]:

$$\int_{-1}^{1} P_m(x)P_n(x)dx=\frac{2}{2n+1}\delta_{mn}, \quad (2)$$

where $\delta_{mn}=\begin{cases}1, m=n,\\ 0, m\neq n\end{cases}$ is the Kronecker delta symbol.

Therefore, the continuous function $F(x)$ in the interval [– 1, 1] can be represented as an infinite linear combination of Legendre polynomials $P_m(x)$ with the corresponding coefficients f_m:

$$F(x)=\sum_0^{\infty} f_m P_m(x), \quad (3)$$

which are calculated as follows [19]:

$$f_m=\int_{-1}^{1} F(x)P_m(x)\frac{2m+1}{2}dx. \quad (4)$$

The sum in formula (3) is limited to a certain number for practical realization. This number should provide the desired error in representing the function $F(x)$ as a finite sum:

$$F_m(x)=\sum_0^{M} f_m P_m(x). \quad (5)$$

Consider the following function

$$F(x)=\exp(i2\pi x) \quad (6)$$

in the interval [– 1, 1]. Fig. 1 shows the real parts of the differences $\mathrm{Re}[F(x)-F_M(x)]$ for $M = 11$, 12, 13, and Fig. 2 shows the imaginary parts of the differences for the same values of M. Analysis of Fig. 1 and Fig. 2 allows to conclude that the function $F(x) = \exp(i2\pi x)$ in the interval [– 1, 1] can be represented with high accuracy as the sum (5) at $M = 13$ (blue curve). Green curves ($M = 15$) almost coincide with the abscissa. The real and imaginary parts of the function $\exp(i2\pi x)$ intersect the abscissa axis 4 times in the interval [– 1, 1]. At the same time, Legendre polynomial $P_{12}(x)$ intersects the abscissa 13 times, namely 3.25 times more than the function.

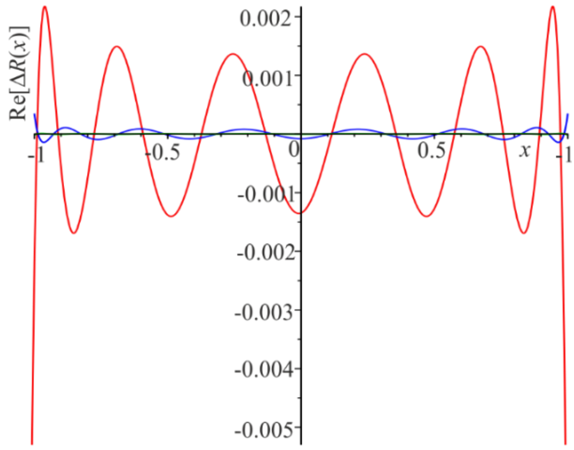

Fig. 1 – Real parts of the differences $\mathrm{Re}[F(x)-F_M(x)]$ for $M = 11$ (red line), $M = 13$ (blue line), $M = 15$ (green line)

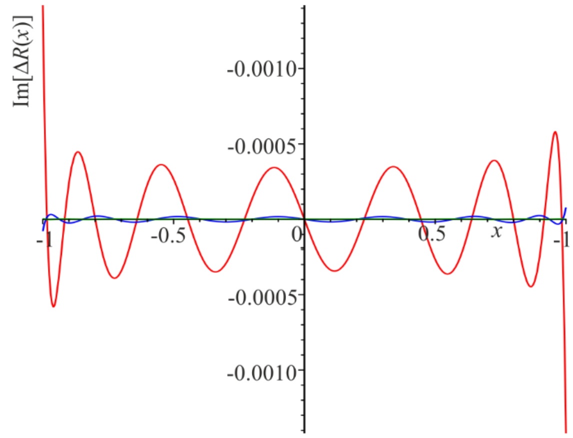

Fig. 2 – Imaginary parts of the differences $\mathrm{Im}[F(x)-F_M(x)]$ for $M = 11$ (red line), $M = 13$ (blue line), $M = 15$ (green line)

If we take the function $F(x) = \exp(i20\pi x)$ (40 intersections with the abscissa axis), then the error will be significantly less than 0.0001 at $M = 81$ in the approximate representation of this function (5). The ratio of the number of intersections with the abscissa axis to the corresponding functions is approximately equal to 2. If $F(x) = \exp(i50\pi x)$ (100 intersections with the abscissa axis), then the first 182 Legendre polynomials are sufficient to represent this function in the form (5) with an error of less than 0.0001. It should be noted that the maximum error in the representation of the functions $F(x)$ by formula (5) is observed at the edges of the interval [– 1, 1], that is at $x = \pm 1$. This is well demonstrated in Fig. 1 and Fig. 2.

Let us analyze what is more rational for filtering with Legendre polynomials in DHI: hologram, amplitude (intensity in digital interferogram) or phase map. We assume that the pixel size in the CCD camera is 10 μm and it is necessary to have at least 2 samples (and even better 5 samples) for one period of the hologram according to the sample theorem [18]. As a result, there will be 400 stripes in the interval of 20 mm. It is necessary to use 800 Legendre polynomials to accurately convey the wave amplitude in the hologram and, accordingly, to filter the noise. That is, hologram filtering requires significant mathematical calculations according to formulas (4) and (5). It is also impractical to filter phase maps with Legendre polynomials, since in phase maps there are gaps of the first kind where the phase jump is equal to π. This will also require a large number of Legendre polynomials for filtering in order to accurately display phase maps.

The number of bands that can be placed in the interval [– 1, 1] will be equal to $N = 2/\Delta l$ for an interferogram obtained with DH at a minimum distance between the bands Δl. Thus, it can be assumed that it is necessary to use $M = 3N = 6/\Delta l$ in order to adequately convey the distribution of amplitudes (intensity) of light in the interference pattern. On the other hand, it is possible to estimate the average distance between zeros of Legendre polynomials in the interval [– 1, 1]. If $M = 35$, then the average distance between zeros will be $2/35 = 0.057$. That is, at $M = 35$, it is possible to transfer the interferogram quite accurately with Legendre polynomials at a minimum distance between the interferogram bands $\Delta l = 6/(M = 0.171)$, $0.171/0.057 \cong 3$.

3. SIGNAL FILTERING USING LEGENDRE POLYNOMIALS

Consider the effect of Legendre polynomials on functions of following types:

$$F_1(x) = \exp(ip\pi x), \qquad (7)$$

$$F_2(x) = \exp\left[ia\cos(p\pi x)\right], \qquad (8)$$

where $i = \sqrt{-1}$, p and a are real numbers.

Function (7) can be related to additive noises that appear, for example, in a photodetector due to current fluctuations in pixels. Function (8) determines multiplicative noises that occur in DHI due to diffuse light scattering from the research object.

The filtering efficiency of functions (7) and (8) for different p using Legendre polynomials can be estimated using the following formula:

$$P_p = \frac{\int_{-1}^{1}\left|F(x)_M\right|^2 dx}{\int_{-1}^{1}\left|F(x)\right|^2 dx} = \frac{\sum_{m=0}^{M} f_m f_m^* \frac{2}{2m+1}}{\int_{-1}^{1}\left|F(x)\right|^2 dx} = \sum_{m=0}^{M} f_m f_m^* \frac{1}{2m+1}. \qquad (9)$$

In (9), it is taken into account that $\int_{-1}^{1}\left|F(x)\right|^2 dx = 2$ if $F(x)$ is defined by formula (7) or (8). If $p = 20$, $M = 35$, then for function (7) $P_{20} = 0.1755$.

Fig. 3 shows the dependences of P_p on p for functions (7) and (8). P_p quickly goes to zero with increasing p for function (7) already at $p > 11$. The picture is slightly different for function (8). In particular, the beginning of a rapid fall depends on a, and there is saturation with increasing a, and the amount of saturation decreases with increasing a.

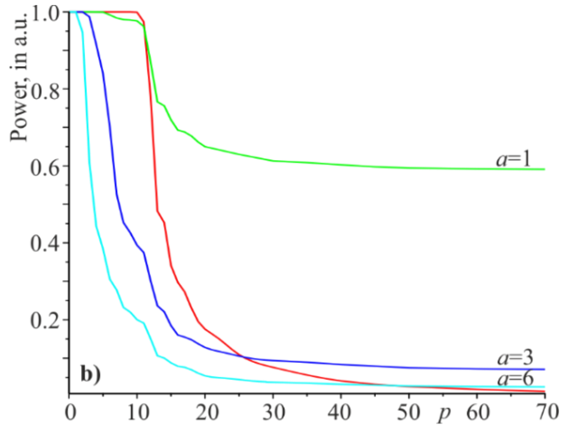

Fig. 3 – Dependences of P_p on p for functions of type (7) (red curve) and for functions of type (8): $a = 1$ (green curve), $a = 3$ (dark blue curve), $a = 6$ (light blue curve)

The noise function can be defined as follows:

$$noise = 0.02\sum_{p=1}^{200}\exp\left[i\left(p\pi x + \phi_p\right)\right]. \qquad (10)$$

The phase φ_p was determined randomly with a uniform distribution on the interval [– π, π] for each harmonic component with index p. The results of numerical simulations showed that the noise strongly distorts the signal.

Fig. 4 shows the difference between the real part of the filtered signal using 13 Legendre polynomials and the real part of the useful signal, which is described by the formula $F_1(x) = \exp(ip\pi x)$. The figure shows that the maximum difference is approximately 0.07. This rather large difference is explained by the fact that the noise contains harmonic components with $p = 1\ldots13$, which are practically not filtered in accordance with Fig. 3 (red curve). We also see a sufficiently large error at the points $x = \pm 1$, which is consistent with Fig. 4. If we take another implementation of random noise, then again we get a large filtering error at the points $x = \pm 1$, and the corresponding difference as a functional dependence on x in the interval [– 1, 1] will change.

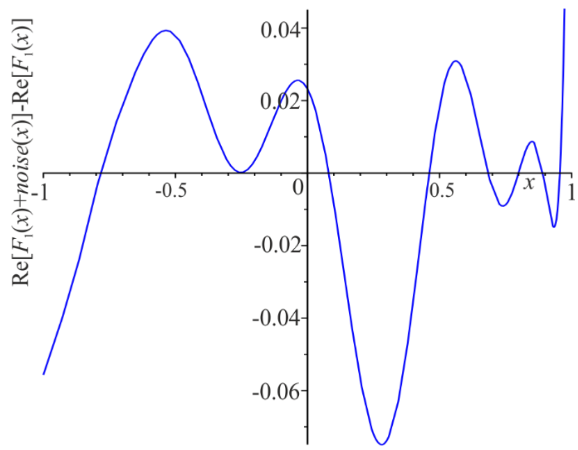

Fig. 4 – Difference between the real part of the filtered signal using 13 Legendre polynomials and the real part of the useful signal described by the formula $F_1(x) = \exp(ip\pi x)$

It should be assumed that the filtering efficiency will improve if the noise signal has harmonic components with a higher frequency than the frequency of the useful signal. The noise function was determined as follows:

$$noise = 0.02\sum_{p=50}^{250}\exp\left[i\left(p\pi x+\phi_p\right)\right]. \quad (11)$$

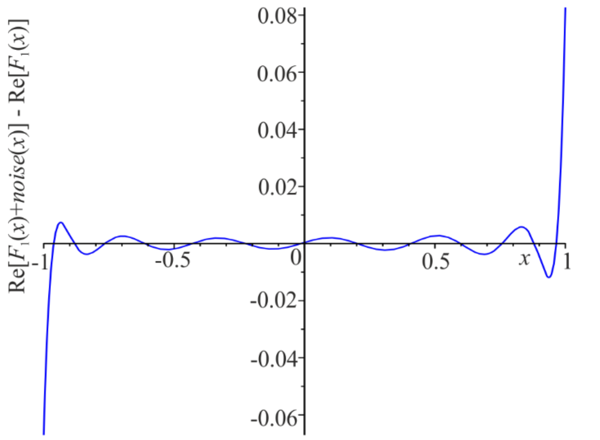

Fig. 5 – Difference between the real part of the filtered signal with noise using 13 Legendre polynomials and the real part of the useful signal described by the formula $F_1(x) = \exp(ip\pi x)$

Fig. 5 shows the difference between the real part of the filtered signal using 13 Legendre polynomials and the real part of the useful signal, which is described by the formula $F_1(x) = \exp(ip\pi x)$ provided that the noise is described by expression (11). We see a significant difference between Fig. 5 and Fig. 4.

The maximum difference is observed at the points $x = \pm 1$. The corresponding difference is less than 0.01 in the interval [– 0.9, 0.9], which is significantly less than the difference in Fig. 5. Such filtering results, when the noise is described by formula (11), are consistent with Fig. 3.

4. CONCLUSIONS

Legendre polynomials, which are orthogonal in the range [– 1, 1], can be effectively used for filtering noisy signals. If for the real or imaginary part of the signal the minimum distance between adjacent local minima (maxima) is equal to Δl, then to approximate such a signal by Legendre polynomials, not less than 13 polynomials are required.

In DHI, it is advisable to use Legendre polynomials for filtering the intensity or amplitude of the field in the interferogram. However, it is unreasonable to filter digital holograms or phase maps of interferograms, as they need to be approximated by a large number of Legendre polynomials.

The filtering efficiency of Legendre polynomials is improved if the noise signal has harmonic components with a higher frequency than the frequency of the useful signal.

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Використання ортогональних поліномів Лежандра для фільтрації зашумлених сигналів на обмеженому інтервалі в просторі координат

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У роботі показано, що ортогональні поліноми Лежандра в інтервалі [– 1, 1] можна ефективно використовувати для фільтрації зашумлених сигналів, у тому числі для фільтрації інтерферограм і фазових карт у цифровій голографічній інтерферометрії. Також з їх допомогою можна ефективно апроксимувати гармонічні сигнали, причому точність апроксимації зростає зі збільшенням кількості використаних поліномів. Фільтрація ґрунтується на використанні оптимальної кількості поліномів Лежандра при апроксимації сигналу. Здійснювати фільтрацію безпосередньо цифрових голограм і фазових карт недоцільно, так як при цьому необхідно використовувати кілька сотень поліномів, що істотно збільшує час чисельних розрахунків. Тому в цифровій голографічній інтерферометрії необхідно фільтрувати безпосередньо амплітуди полів, що розраховуються з цифрової голограми. Інтерферограми та фазові карти можна розрахувати, використовуючи відфільтровані амплітуди полів для різних станів досліджуваного об'єкта. Якщо для дійсної або уявної частини сигналу мінімальна відстань між сусідніми локальними мінімумами (максимумами) дорівнює Δl, то для задовільної апроксимації такого сигналу поліномами Лежандра потрібно $6/\Delta l$ поліномів. Ефективність фільтрації поліномами Лежандра є вищою, якщо шумовий сигнал містить гармонічні складові з частотою більшою за частоту корисного сигналу.

Ключові слова: Цифрова голографія, Цифрова голографічна інтерферометрія, Шум, Фільтрація сигналів, Поліноми Лежандра, Ортогональність поліномів.