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Nonrelativistic Bound State Solutions of the Modified 2D-Killingbeck Potential Involving 2D-Killingbeck Potential and Some Central Terms for Hydrogenic Atoms and Quarkonium System

Abdelmadjid Maireche*

Laboratory of Physics and Material Chemistry, Physics department, Sciences Faculty, University of M'sila, M'sila, Algeria

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In the paper, a new nonrelativistic analytical model is presented and used to modify the Schrödinger equation for atomic scales with two dimensional modified Killingbeck potential (2DMKBP) for the hydrogenic atoms and quarkonium system in noncommutative two-dimensional real space-phase (NC: 2D-RSP). We applied the generalized Bopp's shift method to obtain 2DMKBP involving ordinary 2D-Killingbeck potential (2DKBP) and some central terms proportional to the two infinitesimals parameters $(\theta,\bar{\theta})$. We have also observed the new kinetic term composed of ordinary kinetic term and additive part proportional to infinitesimals parameter $\bar{\theta}$. Furthermore, the global Hamiltonian operator for 2DMKBP involves three fundamental parts: the first one is the ordinary Hamiltonian operator in commutative quantum mechanics (CQM), the second part is the spin-orbit operator $H_{so-k}(r,\theta,\bar{\theta})$, while the third is the modified Zeeman operator $H_{z\text{-k}}(r,\chi,\bar{\sigma})$. Thus, the global energy will include in addition to the ordinary energy in CQM two principal corrections ($E_{d-k}(n,j=l-1/2,l,s)$, $E_{u-k}(n,j=l+1/2,l,s)$) and $E_{z\text{-k}}(n,m,a,b,c)$ corresponding to the spin-orbit interaction and modified Zeeman effect, respectively. The global energy levels $E_{\text{nc-(u-d)clh}}(n,b,c,j,l,s,m)$ depend on discrete atomic quantum numbers $(j,l,s,\ m)$ and the parameters of studied potential $(a,b,\ c)$. Moreover, we have generalized the obtained results to include the quarkonium system ($\gamma(b\bar{b})$ and $\Psi(c\bar{c})$). We have also shown that the mass spectra of quarkonium system were changed to the new forms. The previous results in CQM become special cases when we make the simultaneously two limits $(\theta,\bar{\theta})\to(0,0)$.

Keywords: Schrödinger equation, Killingbeck potential, Noncommutative space–phase, Star product and generalized Bopp's shift method.

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1. INTRODUCTION

The special anharmonic type potential or Killingbeck potential or Cornell plus harmonic potential is a central potential model. It consists of harmonic oscillator plus Cornell potential, which finds applications in many branches of physics including particle physics. Furthermore, it plays an axial role to study the electron and proton system in hydrogen atoms, charmonium system $\Psi(c\bar{c})$ and bottomonium system $\gamma(b\bar{b})$ [1-3]. In 2016, Tapas Das studied this potential in the case of N-dimensional space [3]. The main goal of this work is to apply this study to a particular case of $N=2$ to a new hug symmetry of space and phase, known by noncommutativity of space and phase, based on [4-5], to discover the new spectrum and a possibility to obtain new applications of the modified Killingbeck potential in different fields. In last few years many efforts have been produced to study some potentials using the notions of noncommutativity of space and phase based essentially on Seiberg-Witten map, star product and Bopp's shift method, defined on the first order of two infinitesimal antisymmetric parameters $\left(\theta^{\mu\nu},\bar{\theta}^{\mu\nu}\right)\equiv\frac{1}{2}\varepsilon^{\mu\nu\alpha}\left(\theta_\alpha,\bar{\theta}_\alpha\right)$ as [6-8]:

$$(f*g)(x,p)=(fg)(x,p)-\frac{i}{2}\left(\theta^{\mu\nu}\partial_\mu^x f\partial_\nu^x g+\bar{\theta}^{\mu\nu}\partial_\mu^p f\partial_\nu^p g\right)(x,p). \quad (1)$$

As direct results of the above two modes of star product, the satisfaction of the two new none null commutators, in Schrödinger and Heisenberg pictures, respectively [9-11]:

$$\begin{cases}\left[x^\mu,x^\nu\right]=\left[x^\mu(t),x^\nu(t)\right]=0\\ \left[p^\mu,p^\nu\right]=\left[p^\mu(t),p^\nu(t)\right]=0\end{cases}\Rightarrow$$
$$\begin{cases}\left[\hat{x}^\mu,\hat{x}^\nu\right]_*=\left[\hat{x}^\mu(t),\hat{x}^\nu(t)\right]_*=i\theta^{\mu\nu}\\ \left[\hat{p}^\mu,\hat{p}^\nu\right]_*=\left[\hat{p}^\mu(t),\hat{p}^\nu(t)\right]_*=i\bar{\theta}^{\mu\nu}\end{cases} \quad (2)$$

with $\left[\hat{A}^\mu,\hat{B}^\nu\right]_*\equiv\hat{A}^\mu*\hat{B}^\nu-\hat{B}^\nu*\hat{A}^\mu$. In this work, our aim is twofold. Firstly, in next section, we briefly review the Schrödinger equation with Killingbeck potential in two-dimensional space based on the main refer-

* abmaireche@gmail.com

ence [4] in CQM. In section 3, we discuss the necessary theoretical background that is needed to yield the generalized Bopp's shift method. We also discuss here the procedure to obtain 2DMKBP. Then, we construct the NC spin-orbit Hamiltonian, next, we apply standard perturbation theory to find the spectrum for n^{th} excited states, which are produced automatically by the spin-orbit effect; and we end this section by deducing the spectrum for n^{th} excited states produced automatically by the external magnetic field. In section 4, we resume the global spectrum for 2DMKBP and we conclude the corresponding global NC Hamiltonian in (NC-2D: RSP) symmetries in first order of two infinitesimal parameters θ and $\bar{\theta}$. Secondly, we generalize the obtained results to quarkonium system ($\gamma(b\bar{b})$, $\Psi(c\bar{c})$). In section 5, we calculate the mass spectra of heavy quarkonia in 2D space-phase. Finally, the last section is devoted to the conclusions.

2. REVIEW THE NONRELATIVISTIC BOUND SATE SOLUTION OF THE SCHRÖDINGER EQUATION WITH 2D-KILLINGBECK POTENTIAL

In this section, we shall review the eigenvalues and eigenfunctions for the special anharmonic type potential or Killingbeck potential, which is composed of Cornell plus harmonic potential $V(r)$ [3]:

$$V(r) = ar^2 + br - \frac{c}{r} \tag{3}$$

The parameters a, b and c are constants. The complex eigenfunctions $\Psi(r,\phi)$ in 2D space for above potential satisfy the Schrödinger equation ($\hbar = c = 1$):

$$\left(-\frac{1}{2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\phi^2}\right)+ar^2+br-\frac{c}{r}\right)\times \times\Psi_{n,l,m}(r,\phi) = E_{n,l}\Psi(r,\phi) \tag{4}$$

where $\Psi_{n,l,m}(r,\phi) = R_{nl}(r)exp(im\phi)$, l, μ and $E_{n,l}$ represent the complex wave function, angular momentum and reduced mass of electron and proton system, charmonium system $\Psi(c\bar{c})$ and bottomonium system $\gamma(b\bar{b})$ and the energy, respectively, while $-l \le m \le +l$. The radial part $R_{nl}(r)$ satisfies [3]:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{l^2}{r^2}+2\mu\left(E-ar^2-br+\frac{c}{r}\right)\right]R_{nm}(r) = 0 . \tag{5}$$

The complete normalized wave functions $\Psi_{n,l,m}(r,\phi)$ and corresponding energies $E_{n,l}$ are written as [3]:

$$\Psi_{n,l,m}(r,\phi) = \frac{C_{n,l}}{n!}r^{l+n}\,exp\left(-\sqrt{\frac{\mu a}{2}}r^2 - b\sqrt{\frac{\mu}{2a}}r\right)exp(im\phi) \tag{6}$$

$$E_{n,l} = \sqrt{\frac{a}{2\mu}}(2n+l+2) - \frac{b^2}{4a} \tag{7}$$

with $C_{n,l} = n!\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}^{1/2}$, $\alpha = \sqrt{\frac{\mu a}{2}}$, $\beta = \frac{\mu b}{2\alpha}$ and $\beta\gamma = \beta\frac{n+1}{2\alpha} = \frac{\mu c}{2\alpha}$.

3. THEORETICAL BACKGROUND

3.1 Basic Concept of Generalized Bopp's Shift Method in 2D Space-phase for 2DNKBP

In this section, we shall proceed with writing 2D modified Schrödinger equation (2DMSE) for 2DMKBP in NC: 2D-RSP) symmetries. For this goal, we apply the main steps: we replace ordinary physical quantities: Hamiltonian operator $\hat{H}(p_i, x_i)$, ordinary complex function $\Psi_{nlm}(r,\phi)$ and ordinary energy E_{nl} by new following physical quantities: NC Hamiltonian operator $\hat{H}_{nc-k}(\hat{p}_i, \hat{x}_i)$, new complex function $\Psi\left(\vec{\hat{r}}\right)$ and new energy E_{nc-k}, respectively, in addition, we replace ordinary product by star product (*). Thus, one can rewrite the modified 2DMSE for 2DMKBP as follows [12, 13]:

$$\hat{H}(\hat{p}_i, \hat{x}_i) * \Psi\left(\vec{\hat{r}}\right) = E_{nc-kp}\Psi\left(\vec{\hat{r}}\right). \tag{8}$$

The Hamiltonian operator $\hat{H}_{nc-k}(\hat{p}_i, \hat{x}_i)$ acts by star product on the new complex wave function $\Psi\left(\vec{\hat{r}}\right)$ of the new system to give us the energy eigenvalues E_{nc-k} in (NC: 2D-RSP) symmetries. Next, we will present a brief description about the generalized Bopp's shift method to obtain exact solutions of the 2DMSE. Feeling the fact, we have three general varieties [14-16]:

$$\begin{aligned}
&\hat{H}_{nc-k}(\hat{p}_i, \hat{x}_i) \equiv \hat{H}\left(\hat{p}_i = p_i + \frac{\bar{\theta}_{ij}}{2}x_j ; \hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j\right) && \text{for (NC: 2D-RSP)}\\
&\hat{H}_{nc-k}(\hat{p}_i, \hat{x}_i) \equiv \hat{H}\left(\hat{p}_i = p_i ; \hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j\right) && \text{for (NC: 2D-RS)}\\
&\hat{H}_{nc-k}(\hat{p}_i, \hat{x}_i) \equiv \hat{H}\left(\hat{p}_i = p_i + \frac{\bar{\theta}_{ij}}{2}x_j ; \hat{x}_i = x_i\right) && \text{for (NC: 2D-RP)}
\end{aligned} \tag{9}$$

To find the analytical solutions of the Eq. (8) we must apply the generalized Bopp's shift method instead of solving the 2DMSE for 2DMKBP directly with star product. We treated by using directly the two commutators, in addition to usual commutators on quantum mechanics [22-25]:

$$\begin{cases}\left[\hat{x}^{\mu},\hat{x}^{\nu}\right]_{*}=\left[\hat{x}^{\mu}(t),\hat{x}^{\nu}(t)\right]_{*}=i\theta^{\mu\nu}\\ \left[\hat{p}^{\mu},\hat{p}^{\nu}\right]_{*}=\left[\hat{p}^{\mu}(t),\hat{p}^{\nu}(t)\right]_{*}=i\bar{\theta}^{\mu\nu}\end{cases}\Rightarrow \begin{cases}\left[\hat{x}^{\mu},\hat{x}^{\nu}\right]=\left[\hat{x}^{\mu}(t),\hat{x}^{\nu}(t)\right]=i\theta^{\mu\nu}\\ \left[\hat{p}^{\mu},\hat{p}^{\nu}\right]=\left[\hat{p}^{\mu}(t),\hat{p}^{\nu}(t)\right]=i\bar{\theta}^{\mu\nu}\end{cases} \quad (10)$$

It is well known that the new operators ($\hat{x}_{\mu}$ and $\hat{p}_{\mu}$) are given by the following Darboux transformations [17, 18]:

$$\hat{x}_{\mu}=x_{\mu}-\frac{\theta_{\mu\nu}}{2}p_{\nu} \text{ and } \hat{p}_{\mu}=p_{\mu}+\frac{\bar{\theta}_{\mu\nu}}{2}x_{\nu}. \quad (11)$$

The two variables (x_{μ},p_{μ}) satisfy the usual canonical commutation relations in CQM. In recent work, we are interested in the first variety in Eq. (9). We may go a step further and consider the Bopp's method (modified by a shift), which allows us to reduce the above 2DMSE to ordinary form, in addition two fundamental translations of space and phase which are presenting in Eq. (11):

$$H_{nc-k}(\hat{p}_{i},\hat{x}_{i})\psi(\vec{r})=E_{nc-k}\psi(\vec{r}). \quad (12)$$

The modified Hamiltonian $H_{nc-k}(\hat{p}_{i},\hat{x}_{i})$ that appears above is given by:

$$H_{nc-k}(\hat{p}_{\mu},\hat{x}_{\mu})=\frac{\hat{p}^{2}}{2\mu}+V(\hat{r}). \quad (13)$$

The 2DNKBP (note by $V(\hat{r})$) in the (NC: 2D-RSP) symmetries can be written as:

$$V(\hat{r})=a\hat{r}^{2}+b\hat{r}-\frac{c}{\hat{r}} \quad (14)$$

According to our references [17, 18], we can write the two operators $\hat{r}^{2}$ and $\hat{p}^{2}$ in (NC: 2D-RSP) as follows:

$$\hat{r}^{2}=r^{2}-L_{z}\theta+O(\theta^{2}) \text{ and } \hat{p}^{2}=p^{2}+L_{z}\bar{\theta}+O(\bar{\theta}^{2}). \quad (15)$$

Again, applying the above two relations to the three terms $\left(-\frac{c}{\hat{r}}\right)$, $b\hat{r}$ and $a\hat{r}^{2}$, which will be used to determine the 2DNKBP, gives the following results as:

$$\begin{cases}\frac{c}{r}\to\frac{c}{\hat{r}}=\frac{c}{r}+\frac{c}{2r^{3}}L_{z}\theta+O(\theta^{2})\\ br\to b\hat{r}=br-\frac{b}{2r}L_{z}\theta+O(\theta^{2})\\ ar^{2}\to a\hat{r}^{2}=ar^{2}-aL_{z}\theta+O(\theta^{2})\end{cases} \quad (16)$$

Substitution of Eq. (16) into Eq. (14) gives the 2DMKBP in (NC: 2D-RSP) as follows:

$$V(\hat{r})=V(r)+\left\{\frac{c}{2r^{3}}-\frac{b}{2r}-a\right\}L_{z}\theta \quad (17)$$

Moreover, substitution of Eqs. (17) and (15) into Eq. (13), again leads to a Hamiltonian operator $H_{nc-k}(\hat{p}_{\mu},\hat{x}_{\mu})$ in (NC: 2D-RSP) symmetries as follows:

$$H_{nc-k}(\hat{p}_{\mu},\hat{x}_{\nu})=H(p_{\mu},x_{\nu})+H_{\text{pert-k}}(r,\theta,\bar{\theta}), \quad (18)$$

where $H(p_{\mu},x_{\mu})$ and $H_{\text{pert-k}}(r,\theta,\bar{\theta})$ are given by:

$$H_{pert-k}(r,\theta,\bar{\theta})=\left\{\frac{c}{2r^{3}}-\frac{b}{2r}-a\right\}L_{z}\theta+\frac{L_{z}\bar{\theta}}{2\mu}+O(\theta,\bar{\theta}) \quad (19)$$

and

$$H(p_{\mu},x_{\nu})=\frac{p^{2}}{2\mu}+ar^{2}+br-\frac{c}{r}. \quad (20)$$

It is clear that the operator $H(p_{\mu},x_{\nu})$ is just the Hamiltonian operator for 2DKBP in CQM while the generated part $H_{\text{pert-k}}(r,\theta,\bar{\theta})$ appears because of deformation of noncommutativity space-phase. In the calculations above we have neglected the higher order terms in θ and $\bar{\theta}$ because we are only interested in the corrections of first order.

3.2 2D-spin-orbit Hamiltonian Operators for 2DMKBP in (NC: 2D-RSP) Symmetries

In this subsection, we apply the same strategy, which we have seen in our previous works [17, 18]. Under such particular choice, one can easily reproduce both $L_{z}\theta$ and $L_{z}\bar{\theta}$ to the new physical forms $\gamma\theta\vec{L}\vec{S}$ and $\gamma\bar{\theta}\vec{L}\vec{S}$, respectively, to obtain the new forms of $H_{\text{so-k}}(r,\theta,\bar{\theta})$ for 2DNKBP as follows:

$$H_{\text{so-}k}(r,\theta,\bar{\theta})\equiv\gamma\left\{\left(\frac{c}{2r^{3}}-\frac{b}{2r}-a\right)\theta+\frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S}. \quad (21)$$

Here $\vec{S}$ denotes the spin of fermionic particle like electron and γ is real constant, which can play the role of fine structure constant, thus, spin-orbit interactions $H_{\text{pert-}k}(r,\theta,\bar{\theta})$ appear automatically because of the new properties of space-phase. Now, it is possible to rewrite the above equation as follows:

$$H_{\text{pert-}k}(r,\theta,\bar{\theta})=\frac{\gamma}{2}\left\{\left(\frac{c}{2r^{3}}-\frac{b}{2r}-a\right)\theta+\frac{\bar{\theta}}{2\mu}\right\}\times \times\left(\vec{J}^{2}-\vec{L}^{2}-\vec{S}^{2}\right) \quad (22)$$

where $\vec{J}=\vec{L}+\vec{S}$. It is well known that the eigenvalues

j of the total operator $\vec{J}$ can be obtained from the interval $|l-1/2| \le j \le |l+1/2|$. We have an occasion of determining two-sided bounds to the eigenvalues of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$ as follows:

$$k(j,l,s) \equiv j(j+1)+l(l+1)-s(s+1) = \begin{cases} k_-(j=l-1/2, l, s=1/2) & \text{for spin\_down} \\ k_+(j=l+1/2, l, s=1/2) & \text{for spin\_down} \end{cases} \quad (23)$$

A second occasion of determining a diagonal matrix $H_{so-k}\left(r,\theta,\bar{\theta}\right)$ of order (2×2) with diagonal elements $\left(H_{so-k}\right)_{11}$ and $\left(H_{so-k}\right)_{22}$ as:

$$\begin{aligned} \left(H_{so-k}\right)_{11} &= \gamma k_+\left\{\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\theta+\frac{\bar{\theta}}{2\mu}\right\} \text{if } j=l+\tfrac{1}{2} \\ \left(H_{so-kl}\right)_{22} &= \gamma k_-\left\{\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\theta+\frac{\bar{\theta}}{2\mu}\right\} \text{ if } j=l-\tfrac{1}{2} \end{aligned} \quad (24)$$

After straightforward calculation, one can show that the new radial function $R_{nl}(r)$ satisfies the following differential equation for 2DMKBP:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l^2}{r^2}+2\mu\left(E_{nc-k}-ar^2-br+\frac{c}{r}+k(j,l,s)\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)\theta+k(j,l,s)\frac{\bar{\theta}}{2\mu}\right)\right]R_{nl}(r) \quad (25)$$

In the next parts of this article, we consider the term $H_{\text{pert-k}}\left(r,\theta,\bar{\theta}\right)$ as an infinitesimal part compared to the principal part of the Hamiltonian operator $H(p,x)$ for 2DMKBP in ordinary quantum mechanics. This allows to apply standard perturbation theory to obtain the nonrelativistic energy corrections $E_{k-d}(n,j=l-1/2,l,s)$ and $E_{k-u}(n,j=l+1/2,l,s)$ corresponding to $(j=l-1/2)$ and $(j=l+1/2)$ of hydrogenic atoms at first order of two infinitesimal parameters θ and $\bar{\theta}$.

in (NC: 2D-RSP) Local Symmetries

We are now in a position to attack the main objective of the recent work: to find the nonrelativistic energy corrections $E_{d-k} \equiv E_{d-k}(n,j=l-1/2,l,s)$ and $E_{u-k} \equiv E_{u-k}(n,j=l+1/2,l,s)$ corresponding to $(j=l-1/2)$ and $(j=l+1/2)$ at first order of two parameters θ and $\bar{\theta}$ for hydrogenic atoms for (n,l) states. To achieve this objective, we use time-independent perturbation theory and through the structure constants, which specified the dimensionality of 2DMKBP, thus, we start with following results:

3.3 The Exact Spin-orbit Spectrum for 2DMKBP

$$\begin{aligned} E_{\text{u-k}} &= 2\prod\frac{|C_{n,l}|^2}{n!^2}k_+\int_0^{+\infty} r^{2l+2n+1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)\left\{\theta\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\theta}}{2\mu}\right\}dr \\ E_{\text{u-d}} &= 2\prod\frac{|C_{n,l}|^2}{n!^2}k_-\int_0^{+\infty} r^{2l+2n+1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)\left\{\theta\left(\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\theta}}{2\mu}\right\}dr \end{aligned} \quad (26)$$

A direct simplification gives

$$E_{\text{u-k}} = 2\prod\frac{|C_{n,l}|^2}{n!^2}k_+\left\{\theta\sum_{i=1}^{3}T_i+\frac{\bar{\theta}}{2\mu}T_4\right\} \quad \text{and} \quad E_{\text{d-k}} = 2\prod\frac{|C_{n,l}|^2}{n!^2}k_-\left\{\theta\sum_{i=1}^{3}T_i+\frac{\bar{\theta}}{2\mu}T_4\right\}, \quad (27)$$

where the four terms $T_i\left(i=\overline{1,4}\right)$ are given by:

$$\begin{aligned} T_1 &= \frac{c}{2}\int_0^{+\infty} r^{(2l+2n-1)-1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)dr, \\ T_2 &= -\frac{b}{2}\int_0^{+\infty} r^{(2l+2n)-1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)dr, \\ T_3 &= -a\int_0^{+\infty} r^{(2l+2n+2)-1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)dr, \\ T_4 &= \int_0^{+\infty} r^{(2l+2n+2)-1}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2b\sqrt{\frac{\mu}{2a}}r\right)dr \end{aligned} \quad (28)$$

Now, we apply the following special integration [19]:

$$\int_0^{+\infty} x^{\nu-1}\exp\left(-\beta x^2-\gamma x\right)dx = (2\beta)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad (29)$$

where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$ denotes to the Parabolic cylinder function, $\Gamma(\nu)$ is Gamma function, $Rel(\beta)\rangle 0$ and $Rel(\nu\rangle 0)$. After straightforward calculations, we obtain the results:

$$T_1 = \frac{c}{2}(2\beta)^{-\frac{2n+2l-1}{2}}\Gamma(2n+2l-1)exp\left(\frac{\gamma^2}{8\beta}\right)D_{-(2n+2l-1)}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$
$$T_2 = -\frac{b}{2}(2\beta)^{-\frac{2n+2l+1}{2}}\Gamma(2n+2l+1)exp\left(\frac{\gamma^2}{8\beta}\right)D_{-(2n+2l)}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$
$$T_3 = -a(2\beta)^{-(n+l+1)}\Gamma(2n+2l+2)exp\left(\frac{\gamma^2}{8\beta}\right))D_{-(2n+2l+2)}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad (30$$
$$T_4 = (2\beta)^{-(n+l+1)}\Gamma(2n+2l+2)exp\left(\frac{\gamma^2}{8\beta}\right)D_{-(2n+2l+2)}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$

with $\gamma = 2b\sqrt{\frac{\mu}{2\alpha}}$ and $\beta = 2\sqrt{\frac{\mu a}{2}}$ that allows us to obtain the exact modifications $E_{\text{u-k}}$ and $E_{\text{d-k}}$ of n^{th} states which are produced by spin-orbital effect:

$$E_{\text{u-k}} = \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right)k_+\left\{\theta T_{nc-k} + \bar{\theta}T_{nc-k}\right\}$$
$$E_{\text{d-k}} = \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right)k_-\left\{\theta T_{nc-k} + \bar{\theta}T_{nc-k}\right\} \quad (31)$$

where T_{nc-sk} and T_{nc-pk} are given by:

$$T_{nc-sk} \equiv \sum_{i=1}^{3} T_i \text{ and } T_{nc-pk} \equiv T_4 . \quad (32)$$

The first term T_{nc-sk} is produced from the noncommutative geometry of space while the term T_{nc-pk} is produced from the phase noncommutativity. The following important physical results can be written as:

$$H_{\text{so-k}}(r,\theta,\bar{\theta})R_{nl}(r)exp(im\phi) = \begin{cases} E_{k-d}R_{nl}(r)exp(im\phi) & \text{for } j = l-1/2 \\ E_{k-u}R_{nl}(r)exp(im\phi) & \text{for } j = l+1/2 \end{cases} \quad (33)$$

3.4 The Exact Magnetic Spectrum for 2DMKBP in (NC: 2D-RSP) Local Symmetries

As in the previous subsection, and for further in-depth in the local and global symmetries of (NC: 2D-RSP), it's possible to found another automatically symmetry for 2DNKBP related to the influence of an external uniform magnetic field $\overrightarrow{\aleph}$, if we make the following two transformations simultaneously to ensure that previous calculations are not reputed:

$$(\theta,\bar{\theta}) \to (\chi,\ \bar{\sigma})\ \aleph \ . \quad (34)$$

Here χ and $\bar{\sigma}$ are two new infinitesimal real proportional constants and further insight can be gained when we choose the magnetic field $\overrightarrow{\aleph} = \aleph\vec{k}$, then we can make the following translation:

$$\frac{\gamma}{2}\left\{\theta\left(\frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right\}L_z \to$$
$$\frac{\gamma}{2}\left(\chi\left(\frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\sigma}}{2\mu}\right)\aleph L_z \quad (35)$$

Let us introduce the modified magnetic Hamiltonian operator $H_{\text{z-k}}(r,\chi,\ \bar{\sigma})$ for 2DNKBP in (NC: 2D-RSP) symmetries as:

$$H_{\text{z-k}}(r,\chi,\ \bar{\sigma}) = \frac{\gamma}{2}\left(\chi\left(\frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\sigma}}{2\mu}\right)\left(\overrightarrow{\aleph}\vec{J} - \vec{S}\overrightarrow{\aleph}\right) \quad (36)$$

Here $\left(-\vec{S}\overrightarrow{\aleph}\right)$ denotes the ordinary Hamiltonian of Zeeman effect for 2DKBP in QM, it is a profound feature of both QM and NCQM. To obtain the exact NC magnetic modifications of energy $E_{\text{z-k}}(n,m)$ for hydrogenic atoms for (n, l) states for 2DMKBP, we replace both $k_+(j = l+1/2, l, s)$ and θ in the Eq. (33) by the new discreet quantum number m and the infinitesimal parameter χ, respectively:

$$E_{\text{z-k}}(n,m,a,b,c) = \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)} \times$$
$$\times exp\left(-\frac{\beta^2}{2\alpha}\right)m\left\{\chi T_{nc-sk} + \bar{\sigma}T_{nc-pk}\right\}\aleph \quad (37)$$

However, very little has been achieved in the solution of MSE for 2DMKBP. Thus, we can conclude immediately:

$$H_{\text{z-k}}(r,\chi,\ \bar{\sigma})\Psi_{nlm}(r,\phi) = \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right) \quad m\left\{\chi T_{nc-sk} + \bar{\sigma}T_{nc-pk}\right\}\aleph\Psi_{nlm}(r,\phi). \quad (38)$$

4. RESULTS AND DISCUSSION OF GLOBAL SPECTRUM FOR 2DMKBP IN GLOBAL (NC: 2D-RSP) SYMMETRIES

We have solved the modified radial Schrödinger equation and obtained the differences in the energy eigenvalues $E_{enc-(d,u)}(n, j = l \pm 1/2, l, s = 1/2)$ for the 2DNKBP in equations (33) and (39) which are produced automatically by the effects of spin-orbit interaction $H_{\text{pert-k}}(r,\theta,\bar{\theta})$ and new Zeeman effect $H_{\text{z-k}}(r,\chi,\ \bar{\sigma})$, respectively. Now, we wish to summarize the obtained results of the previous subsections for hydrogenic atoms moving in 2DMKBP. According to three equations (7), (31) and (37), the explicit bound state energies for 2DMKBP take the form for hydrogenic atoms:

$$\begin{aligned} E_{\text{nc -d}}(n,l,j,s,m) &= \sqrt{\frac{a}{2\mu}}(2n+l+2) - \frac{b^2}{4a} + \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)} exp\left(-\frac{\beta^2}{2\alpha}\right)\left(k_-\left\{\theta T_{nc-k} + \bar{\theta} T_{nc-k}\right\} + m\left\{\chi T_{nc-sk} + \bar{\sigma} T_{nc-pk}\right\}\aleph\right) \\ E_{\text{nc -u}}(n,l,j,s,m) &= \sqrt{\frac{a}{2\mu}}(2n+l+2) - \frac{b^2}{4a} + \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)} exp\left(-\frac{\beta^2}{2\alpha}\right)\left(k_+\left\{\theta T_{nc-k} + \bar{\theta} T_{nc-k}\right\} + m\left\{\chi T_{nc-sk} + \bar{\sigma} T_{nc-pk}\right\}\aleph\right) \end{aligned} \quad (39)$$

Thus, the obtained lower and upper bound energy levels $E_{\text{nc-d}}(n,l,j,s,m)$ and $E_{\text{nc -u}}(n,l,j,s,m)$ for hydrogenic atoms for (n, l) states are the sum of the principal part of energy $E_{n,l}$ and the two energy corrections $E_{k-d}(n, j = l-1/2, l, s)$, $E_{k-u}(n, j = l+1/2, l, s)$ and $E_{\text{z-k}}(n,m)$, this is one of the main motivations for the topic of this work. It is clear, that the obtained eigenvalues of energies are real, which allow us to consider the NC diagonal Hamiltonian $H_{nc-k}(r,\theta,\bar{\theta},\chi,\bar{\sigma})$ as a Hermitian operator. In addition and regarding the previous obtained results (20), (21) and (36), the global Hamiltonian operator, at first order in θ and $\bar{\theta}$ with 2DNKBP for hydrogenic atoms for (n, l) states takes the form as:

$$H_{nc-k}(r,\Theta,\bar{\theta},\chi,\bar{\sigma}) = \left(\frac{\Delta}{2\mu} - \frac{Ze^2}{r} + ar^2 + br - \frac{c}{r}\right) + \gamma\left(\left(\frac{c}{2r^3} - \frac{b}{2r} - a\right)\theta + \frac{\bar{\theta}}{2\mu}\right)\mathbf{L}\vec{S} + \left(\left(\frac{c}{2r^3} - \frac{b}{2r} - a\right)\chi + \frac{\bar{\sigma}}{2\mu}\right)\left(\aleph\vec{J} - \vec{S}\aleph\right). \quad (40)$$

This is the equation for hydrogenic atoms under the influence of modified 2D-Killingbeck interactions. It should be pointed out that this treatment considers only first order terms in either θ or $\bar{\theta}$. Clearly, the first part in Eq. (40) presents the Hamiltonian operator in the ordinary quantum mechanics for 2D-Killingbeck potential, while the second and the third parts respectively present the spin-orbit and modified Zeeman Hamiltonian operators for 2DMKBP, which are induced automatically by the NC properties of space and phase. Thus, the important result from this work is:

$$H_{nc-k}(r,\theta,\bar{\theta},\chi,\bar{\sigma})\Psi_{nlm}(r,\phi) = \begin{cases} \left\{E_{nl} + E_{\text{so-d}}(n,j,l,s) + E_{\text{z-k}}(n,m)\right\}\Psi_{nlm}(r,\phi) & \text{for}: j = l-1/2 \\ \left\{E_{nl} + E_{\text{so-u}}(n,j,l,s) + E_{\text{z-k}}(n,m)\right\}\Psi_{nlm}(r,\phi) & \text{for}: j = l+1/2 \end{cases} \quad (41)$$

It is evident to consider the atomic quantum number m can take $(2l+1)$ values and we have also two values for $j = l \pm \frac{1}{2}$ corresponding to up and down polarities, thus, every state in usual 2D-space of energy for 2DMKBP will be $2(2l+1)$ sub-state in (NC: 2D-RSP). Thus, the total complete degeneracy of obtained energy level of the new 2DMKBP potential is obtained as a sum of all allowed values of l. Total degeneracy is thus

$$\sum_{i=0}^{n-1} 2(2l+1) \equiv 2n^2. \quad (42)$$

In the limit $(\theta,\bar{\theta}) \to (0,0)$ we have:

$$\lim_{(\theta,\bar{\theta})\to(0,0)} E_{\text{nc-(d,u)}}(n,l,j,s,m) = E_{nl}. \quad (43)$$

This is the result for the ordinary 2DKBP in ordinary quantum mechanics [4]. Now, it is possible to generalize the energy eigenvalues $E_{\text{nc -(u,d)}}(n,l,j,s,m)$ to case of spin $s \neq 1/2$, we replace $k_+(j,l,s=1/2)$ by the generalized values $k(j,l,s) \equiv j(j+1) + l(l+1) - s(s+1)$. Thus, we have:

$$E_{nc-k}(n,l,j,s,m) = \sqrt{\frac{a}{2\mu}}(2n+l+2) - \frac{b^2}{4a} + \frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)} exp\left(-\frac{\beta^2}{2\alpha}\right)\left(k(j,l,s)\left\{\theta T_{nc-k} + \bar{\theta} T_{nc-k}\right\} + m\left\{\chi T_{nc-sk} + \bar{\sigma} T_{nc-pk}\right\}\aleph\right). \quad (44)$$

We now look at some special cases and relationships between our recent results and some other existing results in our previous works.

Case 1: First case, when we set $(a = b = 0)$ and $c = ze^2$, the 2DMKBP reduces to the modified Coulomb potential, it is easy to show that equations (19), (39) and (44) are reduced to the modified interaction $H_{pert-col}(r,\theta,\bar{\theta})$ of a particle in the modified Coulomb potential and corresponding NC spectrum $E_{\text{nc-(u-d)}}$, respectively.

$$H_{pert-k}(r,\theta,\bar{\theta}) \to H_{pert-col}(r,\theta,\bar{\theta}) \equiv -\frac{ze^2}{2r^3}L_z\theta + \frac{L_z\bar{\theta}}{2\mu}. \quad (45)$$

$$E_{nc-u}\left(n,l,j,s,m,a=b=0,c=ze^2\right)=-\frac{Z^2\mu e^4}{2(n+l+1)}+E_{u-k}\left(a=b=0,c=ze^2\right)+E_{z-k}\left(n,m,a=b=0,c=ze^2\right)$$
$$E_{nc-d}\left(n,l,j,s,m,a=b=0,c=ze^2\right)=-\frac{Z^2\mu e^4}{2(n+l+1)}+E_{d-k}\left(a=b=0,c=ze^2\right)+E_{z-k}\left(n,m,a=b=0,c=ze^2\right) \quad (46)$$

Case 2: Similarly, if we set $b=c=0$ and $a=\frac{1}{2}\mu\omega$, equations (19) and (39) are reduced to the results of modified Harmonic oscillator $H_{pert-oh}\left(r,\theta,\bar{\theta}\right)$ and corresponding modified bound state energy spectrum $E_{nc-oh}\left(n,l,j,s,m,b=c=0,a=1/2\mu\omega^2\right)$ of a vibrating rotating diatomic molecule, respectively.

$$H_{pert-oh}\left(r,\theta,\bar{\theta}\right)=-aL_z\theta+\frac{\vec{L}\vec{\bar{\theta}}}{2\mu}+O\left(\theta,\bar{\theta}\right) \quad (47)$$

and

$$E_{nc-oh}\left(n,l,j,s,m,b=c=0,a=1/2\mu\omega^2\right)=\omega(n+l+3/2)+E_{(u-d)-k}\left(b=c=0,a=1/2\mu\omega^2\right)+E_{z-k}\left(n,m,b=c=0,a=1/2\mu\omega^2\right). \quad (48)$$

5. MASS SPECTRA OF HEAVY QUARKONIA IN 2D SPACE-PHASE

In this section, the mass spectra of quarkonium system ($\gamma\left(b\bar{b}\right),\Psi\left(c\bar{c}\right)$) properties of $\left(b\bar{b}\right)$ and $\left(c\bar{c}\right)$ mesons are calculated, in which the quarkonium meson have quark and antiquark masses. The following relation is used for determining quarkonium masses in the 2D space-phase:

$$M=m_q+m_{\bar{q}}+E_{nl}\to M_{new}= =m_q+m_{\bar{q}}+E_{nc-k}\left(n,l,j,s,m\right) \quad (49)$$

By substituting Eq. (7) and Eq. (44) into Eq. (49), the variation in quarkonium mass δM in 2D space takes the following form:

$$M_{new}=M+\frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right)\left(k(j,l,s)\left\{\theta T_{nc-k}+\bar{\theta}T_{nc-k}\right\}+m\left\{\chi T_{nc-sk}+\bar{\sigma}T_{nc-pk}\right\}\aleph\right) \quad (50)$$

where M is quarkonium system mass in the case of ordinary quantum mechanics as follows:

$$M=m_q+m_{\bar{q}}+E_{nl} \quad (51)$$

Thus, the properties of noncommutative space phase correct the quarkonium system by:

$$\delta M\equiv M_{new}-M=\frac{4\prod(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma(l+n+1)}exp\left(-\frac{\beta^2}{2\alpha}\right)\left(k(j,l,s)\left\{\Theta T_{nc-k}+\bar{\theta}T_{nc-k}\right\}+m\left\{\chi T_{nc-sk}+\bar{\sigma}T_{nc-pk}\right\}\aleph\right). \quad (52)$$

Regarding, our obtained results for global Hamiltonian operator and corresponding eigenvalues, which contained two important physical phenomena's, we can decelerate, the high precision measurements in quantum mechanical systems may be able to reveal the noncommutativity of space and phase.

6. CONCLUSIONS

In this paper, the energy levels $E_{\text{nc-d}}\left(n,j,l,s,m\right)$ and $E_{\text{nc-u}}\left(n,j,l,s,m\right)$ of hydrogenic atoms for (n, l) states have been examined analytically under 2DMKBP (in the case of NC: 2D-RSP) via the generalized Bopp's method and standard perturbation theory. We briefly summarize what has been achieved in this reach work and comment on the outlook on future work that can follow from this paper:

- We have reviewed the nonrelativistic 2D-Killingbeck potential for hydrogenic atoms and the Bopp's shift method.
- We have solved the 2DMSE for its bound states with 2DMKBP.
- Our approach allows us to re-derive the global Hamiltonian operators $H_{nc-k}\left(r,\theta,\bar{\theta},\chi,\bar{\sigma}\right)$ (which contain of two new perturbative terms: the first one is spin-orbit interaction $\hat{H}_{so-k}\left(r,\theta,\bar{\theta}\right)$ while the second term is known by modified Zeeman effect $\hat{H}_{z-k}\left(r,\chi,\bar{\sigma}\right)$) and corresponding new energy eigenvalues corresponding to up and down polarities $E_{\text{nc-d}}\left(n,l,j,s,m\right)$ and $E_{\text{nc-u}}\left(n,l,j,s,m\right)$, respectively.
- Modified mass spectra of charmonium system $\Psi\left(c\bar{c}\right)$ and bottomonium system $\gamma\left(b\bar{b}\right)$ are well studied using Eq. (50).
- We hope to get some interesting new applications to this new potential in the study of different fields of matter sciences, because our results are not only interesting for the pure theoretical physicists, but also for experimental physicists (various fields of physics, such as nuclear physics, condensed matter physics, atomic physics and quantum chromodynamics).

- The noncommutative solution of 2DMKBP for hydrogenic atoms can be reduced into the Schrödinger solution under the usual limit $(\theta,\bar{\theta}) \to (0,0)$. This suggests that the obtained new results are correct.
- We recommend applying the generalized Bopp's method and standard perturbation theory in finding bound state solutions of some other model potentials in NCQM.

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Розв'язки нерелятивістських граничних станів модифікованого 2D потенціалу Кіллінгбека, що включає 2D потенціал Кіллінгбека і деякі центральні доданки для гідрогенних атомів і кварконієвої системи

Abdelmadjid Maireche

Laboratory of Physics and Material Chemistry, Physics Department, Sciences Faculty, University of M'sila, M'sila, Algeria

У роботі представлена нова нерелятивістська аналітична модель, яка використовується для модифікації рівняння Шредінгера для атомних масштабів з двовимірним модифікованим потенціалом Кіллінгбека (2DMKBP) для гідрогенних атомів і кварконієвої системи в некомутативній двомірній реальній просторовій фазі (NC: 2D-RSP). Ми застосували узагальнений метод зсуву Боппа, щоб отримати 2DMKBP, що включає звичайний двовимірний потенціал Кіллінгбека (2DKBP) і деякі центральні доданки пропорційні двом нескінченно малим параметрам $(\theta,\bar{\theta})$. Ми також спостерігали новий кінетичний доданок, що складається із звичайного кінетичного доданку і адитивної частини, пропорційної нескінченно малому параметру $\bar{\theta}$. Крім того, глобальний оператор Гамільтона для 2DMKBP включає три фундаментальні частини: перша – це звичайний оператор Гамільтона в комутативній квантовій механіці (CQM), друга частина – спин-орбітальний оператор $H_{so-k}(r,\theta,\bar{\theta})$, а третя – модифікований оператор Зеемана $H_{z\text{-}k}(r,\chi,\bar{\sigma})$. Таким чином, загальна енергія буде включати крім звичайної енергії в CQM дві головні поправки ($E_{d-k}(n,j=l-1/2,l,s)$, $E_{u-k}(n,j=l+1/2,l,s)$) і $E_{z\text{-}k}(n,m,a,b,c)$, що відповідає спін-орбітальній взаємодії і модифікованому ефекту Зеємана відповідно. Основні енергетичні рівні $E_{nc\text{-}(u\text{-}d)clh}(n,b,c,j,l,s,m)$ залежать від дискретних атомних квантових чисел $(j,l,s,\ m)$ та параметрів $(a,b,\ c)$ потенціалу, що розглядається. Крім того, отримані результати узагальнено з урахуванням кварконієвої системи (($\gamma(b\bar{b})$ та $\Psi(c\bar{c})$). Ми також показали, що мас-спектри кварконієвої системи були змінені на нові форми. Попередні результати в CQM стають особливими випадками, коли ми одночасно використовуємо два обмеження $(\theta,\bar{\theta}) \to (0,0)$.

Ключові слова: Рівняння Шредінгера, Потенціал Кіллінгбека, Некомутативний простір-фаза, Метод Star, Узагальнений метод зсуву Боппа.