Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics

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In this recently work, we have obtained the analytical solutions of the modified Schrödinger equation (MSE) with new Coulomb potentials plus linear and harmonic radial terms (NCPLHRT) for hydrogenic atoms in noncommutative 3-dimensional real space-phase (NC: 3D-RS). We applied, the generalized Bopp’s shift method and standard perturbation theory in the framework of two infinitesimal parameters \( \{ \Theta, \delta \} \) due to (space-phase) noncommutativity, we obtained new energy eigenvalues \( E_{nc\text{-}	ext{H}(\lambda, g, Z, j, l, s, m)} \), which depended with discrete atomic quantum numbers \( (j, l, s, m) \) and parameters of studied potential \( (\lambda, g, Z) \), in addition to the corresponding new Hamiltonian operator \( H_{nc\text{-}	ext{H}}(\lambda, g, Z) \). Our research also conveys another innovative feature, the global group symmetry (NC: 3D-RS) were broken simultaneously and replaced by the residual local subgroup (NC: 3D-RS) under interaction of hydrogenic atoms with NCPLHRT.

Keywords: Schrödinger equation, Coulomb potentials plus linear and harmonic radial terms, Noncommutative space-phase, Star product and Generalized Bopp’s shift method.

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1. INTRODUCTION

Generally, throughout the literature, the Coulomb potentials plus linear and harmonic radial terms (CPLHRT) of the form

\[ V(r) = -\frac{Z}{r} + gr + \lambda r^2 \]

have attracted great interest in atomic and molecular physics and quantum chromodynamics, this potentials supposed to be responsible for the interaction between quark and antiquark and alternatingly, it is called Killingbeck or Cornell plus harmonic potential [1–3]. The aim of our work is to extended, the study of Enrique Castro and Pablo Martin [1] to the case of extended quantum mechanics (EQM), or noncommutative quantum mechanics (NCQM) to finding other new applications and more profound interpretations in the subatomic scales. On the other hand, we extend our study of [4] from noncommutative two dimensional real spaces (NC: 2D-RS) to the case of three-dimensional real space-phase (NC: 3D-RS). We based on previous studies of other authors and some of our related works in this context. The EQM known by noncommutativity of space-time, introduced firstly by Heisenberg, and formalized by Snyder at 1947, suggest by the physical recent results in string theory. The nonrelativistic energy levels for hydrogenic atoms, which interacted with new Coulomb potentials plus linear and harmonic radial terms (NCPLHRT) in the context of NC space, have not been obtained yet. In last few years many effort has been produced to study some potentials using the notions of noncommutativity of space and phase based essentially on Seiberg-Witten map and generalized Bopp’s shift method and the star product, defined on the first order of two infinitesimal parameters antisymmetric

\[ 2(\Theta^{\mu\nu}, \delta^{\mu\nu}) = \varepsilon^{\mu\nu\rho\sigma}(\Theta^\rho_\mu, \delta^\rho_\mu) \] as [5–9]:

\[ (f * g)(x, p) = (fg)(x, p) - \frac{i}{2} (\Theta^{\mu\nu} \partial_\mu \phi(x, p) \partial_\nu \phi(x, p)) \] (1)

As direct results for the above two modes of star product due to the space-space and phase-phase noncommutativity, allow us to finding new none nulls commutators [10–14]:

\[ \left[ x^\mu, x^\nu \right] = i\Theta^{\mu\nu} \quad \text{and} \quad \left[ p^\mu, p^\nu \right] = i\delta^{\mu\nu} \] (2)

The present paper is arranged as follows. Section 2 is a brief outline of the ordinary SE with NCPLHRT on based to the main ref. [1]. The Section 3 is devoted to studying the MSE by applying the generalized Bopp’s shift method and standard perturbation theory we find the quantum spectrum of \( n^{th} \) excited levels for modified spin-orbital interaction in the framework of the global quantum group (NC–3D–RS) for NCPLHRT, then we end this section by derive the magnetic spectrum for NCPLHRT. In the fourth section, we resume the global spectrum and corresponding NC Hamiltonian operator for NCPLHRT and corresponding energy levels of hydrogenic atoms. Conclusion of the work is placed in the last section.

2. REVIEW THE EIGENVALUES OF THE SCHRÖDINGER EQUATION WITH CPLHRT

Let us begin this section by reviewing the nonrelativis-
istic quantum description of an atom or molecule with an induced by spherically symmetric potential that has the form [1]:

\[ V(r) = -Z/r + gr + \lambda r^2 \] (3)

The parameters \(Z, g(E_0e^{-a_0^2})\) and \(\lambda\) are atomic number, the electric field (in atomic units) and a positive constant, respectively. The second term in the above equation corresponds to the scalar potential proportional to the radial distance. The Hamiltonian operator for hydrogenic atoms with a polynomial perturbation is [1]:

\[ \hat{H}_0 = \hat{H}_0(r) + V_1(r) \] (4)

The principal Hamiltonian operator \(H_0\) and the perturbed potential \(V_1\) as [1]:

\[ H_0 = -\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \lambda r^2 + \frac{L^2}{2r^2} \]

\[ V_1 = -Z/r + gr \]

where \(L\) is the angular momentum operator. The complex eigenfunctions \(\Psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)\) in 3-dimensional space while the radial part \(R_{nl}(r)\) satisfies (Here we use atomic units \(\hbar = m_e = c = 1\)) [1]:

\[ \left[ \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + 2 \left( E_{nl} + Z/r + gr - \lambda r^2 \right) \right] \times R_{nl}(r) = 0 \] (5)

The purpose of the present paper is to attempt study the MSE with NCPLHRT potential (see below) in (NC: 3D-RSP) symmetries using the generalized Bopp shift method which depend on the concepts that we present below in the third section to discover the new symmetries and a possibility to obtain another applications to this potential in different fields.

3. METHOD AND THEORETICAL APPROACH

In this section, we shall give an overview or a brief preliminary for NCPLHRT, in (NC: 3D-RSP) symmetries. To perform this task the physical form of modified Schrödinger equation (MSE), it is necessary to replace ordinary three-dimensional Hamiltonian operators \(\hat{H}(p_i,x_i)\) by new three Hamiltonian operators \(\hat{H}_{nc-dh}\) and ordinary complex wave function \(\Psi(r)\) by new complex wave function \(\Psi(\hat{r})\) in new values \(E_{nc-dh}\), respectively. In addition to replace the ordinary old product by new star product (*), which allow us to constructing the MSE in (NC-3D: RSP) symmetries as [16-19]:

\[ \hat{H}_{nc-dh}(p_i,x_i)\Psi(\hat{r}) = E_{nc-dh}\Psi(\hat{r}) \Rightarrow \hat{H}(\hat{p}_i,\hat{x}_i)\Psi(\hat{r}) = E_{nc-dh}\Psi(\hat{r}) \] (12)

The Bopp’s shift method employed in the solutions enables us to explore an effective way of obtaining the NCPLHRT in EQM, it based on the following new commutators: [16-19]:

\[ [x_i,x_j] = 0 \rightarrow [\hat{x}_i,\hat{x}_j] = [\hat{x}_i(t),\hat{x}_j(t)] = i\hbar \delta_{ij} \]

\[ [\hat{p}_i,\hat{p}_j] = 0 \rightarrow [\hat{p}_i,\hat{p}_j] = [\hat{p}_i(t),\hat{p}_j(t)] = i\hbar \delta_{ij} \] (13)

The new generalized positions and momentum coor-
\[
\left( \hat{x}_i, \hat{p}_i \right) \Rightarrow \left( \hat{\tilde{x}}_i, \hat{\tilde{p}}_i \right) = \left( x_i - \frac{\theta}{2} p_x + \frac{\theta}{2} x_i \right)
\]
(14)

The above equation allows us to obtain the two operators \((r^2, p^2)\) in (NC: 3D-RSP) respectively [20-22]:
\[
\left( r^2, p^2 \right) \Rightarrow \left( \hat{\tilde{r}}^2, \hat{\tilde{p}}^2 \right) = \left[ r^2 - L\hat{\Theta}, p^2 + \tilde{L}\hat{\Theta} \right]
\]
(15)

The two couplings \(L\hat{\Theta}\) and \(\tilde{L}\hat{\Theta}\) are \((\Theta_1 + \Theta_22 + \Theta_31)\) and \((\Theta_1 + \Theta_22 + \Theta_31)\), respectively and \((L_1, L_2, L_3)\) are the three components of angular momentum \(L\) while the new parameter \(\Theta_0\) equal \(\theta/2\). Thus, the reduced Schrödinger equation (without star product) can be written as:
\[
\hat{\mathcal{H}}(\tilde{p}_i, \tilde{x}_i) \ast \Psi(\tilde{r}) = E_{nc-ch} \Psi(\tilde{r})
\]
(16)

\[
\Rightarrow \hat{\mathcal{H}}_{nc-ch}(\tilde{p}_i, \tilde{x}_i) \Psi(\tilde{r}) = E_{nc-ch} \Psi(\tilde{r})
\]

The new operator of Hamiltonian \(\hat{\mathcal{H}}_{nc-ch}(\tilde{p}_i, \tilde{x}_i)\) can be expressed as:
\[
\hat{\mathcal{H}}_{nc-ch}(\tilde{p}_i, \tilde{x}_i) = \frac{\hat{\tilde{\rho}}^2}{2} + V_{ch}(\tilde{r})
\]
(17)

The NCPLHRT \(V_{ch}(\tilde{r})\) is given by:
\[
V_{ch}(\tilde{r}) = -Z / \tilde{r} + \frac{g\tilde{r}}{2} + \lambda \tilde{r}^2
\]
(18)

After straightforward calculations, we can obtain the important terms \((-Z / \tilde{r}, g\tilde{r}^2\) and \(\lambda \tilde{r}^2\), which will be used to determine the NCPLHRT in (NC: 3D-RSP) symmetries as:
\[
-\frac{Z}{r} \rightarrow -\frac{Z}{\tilde{r}} = -\frac{Z}{r} \frac{\tilde{\Theta}}{2r} + O(\Theta)
\]
\[
gr \rightarrow g\tilde{r}^2 = gr - \frac{g}{2r} \tilde{L}\hat{\Theta} + O(\Theta)
\]
\[
\lambda r^2 \rightarrow \lambda\tilde{r}^2 = \lambda r^2 - \lambda \tilde{L}\hat{\Theta} + O(\Theta)
\]
(19)

We further the equations (19) and (15) into Equations (17) we obtained the global our working new Hamiltonian operator \(\hat{\mathcal{H}}_{nc-ch}(\tilde{r})\) for NCPLHRT satisfies the equation in (NC: 3D-RSP) symmetries:
\[
\hat{\mathcal{H}}_{nc-ch}(\tilde{r}) = \hat{\mathcal{H}}_{ch}(p_i, x_i) - \left( \frac{Z}{2\tilde{r}} + \frac{g}{2r} + \lambda \right) \tilde{L}\hat{\Theta} + \frac{\tilde{L}\hat{\Theta}}{2}
\]
(20)

where the operator \(\hat{\mathcal{H}}_{ch}(p_i, x_i)\) is just the ordinary Hamiltonian operator for CPLHRT in commutative space:
\[
\hat{\mathcal{H}}_{ch}(p_i, x_i) = \frac{p_i^2}{2} - Z / \tilde{r} + \frac{g}{2r} \lambda r^2
\]
(21)

while the rest four terms are proportional to two infinitesimals parameters \((\Theta \hat{\Theta})\) and then we can consider as a perturbations terms \(\hat{\mathcal{H}}_{per-ch}(r)\) in (NC: 3D-RSP) symmetries for NCPLHRT as:
\[
\hat{\mathcal{H}}_{per-ch}(r) = -\left( \frac{Z}{2\tilde{r}} + \frac{g}{2r} + \lambda \right) \tilde{L}\hat{\Theta} + \frac{\tilde{L}\hat{\Theta}}{2}
\]
(22)

3.1 The Exact Modified Spin-Orbital Spectrum for NCPLHRT in Global (NC: 3D- RSP) Symmetries

In this subsection, we apply the same strategy, which we have seen in our previously works [19-22], under such particular choice, one can easily reproduce both couplings \((\Theta \hat{\Theta} \text{ and } \tilde{L}\hat{\Theta})\) to the new physical forms \((\Theta \hat{\Theta} \hat{\Theta} \text{ and } \tilde{L}\hat{\Theta} \hat{\Theta})\), respectively, to obtain the new forms of \(\hat{\mathcal{H}}_{nc-ch}(r, \Theta, \tilde{\Theta})\) for 3D-NCPLHRT as follows:
\[
\hat{\mathcal{H}}_{nc-ch}(r, \Theta, \tilde{\Theta}) = -\gamma \left( \frac{Z}{2\tilde{r}} + \frac{g}{2r} + \lambda \right) \Theta + \frac{\tilde{\Theta}}{2} \hat{\Theta} \hat{\Theta} \hat{\Theta} \hat{\Theta}
\]
(23)

Here \(\gamma = \frac{1}{137}\) is a new constant, which play the role of fine structure constant, we have chosen the two vectors \((\Theta \text{ and } \tilde{\Theta})\) parallel to the spin \(\hat{\Theta}\) of hydrogenic atoms. Furthermore, the above perturbative terms \(\hat{\mathcal{H}}_{per-ch}(r)\) can be rewritten to the following new form:
\[
\hat{\mathcal{H}}_{nc-ch}(r, \Theta, \tilde{\Theta}) = -\gamma \left( \frac{Z}{2\tilde{r}} + \frac{g}{2r} + \lambda \right) \Theta + \frac{\tilde{\Theta}}{2} \hat{\Theta} \hat{\Theta} \hat{\Theta} \hat{\Theta}
\]
(24)

With \(G^2 = \hat{J} - \hat{L} \cdot \hat{S} \cdot \hat{S}\), this operator traduces the coupling between spin \(\hat{\Theta}\) and orbital momentum \(\hat{\Theta} \hat{\Theta} \hat{\Theta} \hat{\Theta}\). The set \((\hat{\mathcal{H}}_{nc-ch}(r, \Theta, \tilde{\Theta}), J_z^2, S^2, \hat{J}_z)\) forms a complete of conserved physics quantities and for \(\hat{\Theta} = 1/2\), the eigenvalues of the spin orbital coupling operator are \(k_z = \frac{1}{2} \left( \frac{l_z + 1}{2} \pm 1 + l_z \right) \pm \frac{3}{4}\) corresponding:
\[
j = l + 1/2 \text{ (spin up) and } j = l - 1/2 \text{ (spin down)}, respectively, then, one can form a diagonal \((3 \times 3)\) matrix, with diagonal elements are \((\hat{\mathcal{H}}_{nc-ch})_{11}, (\hat{\mathcal{H}}_{nc-ch})_{22}\) and \((\hat{\mathcal{H}}_{nc-ch})_{33}\) for NCPLHRT in (NC: 3D-RSP) symmetries, as:

After profound calculation, one can show that, the
new radial function $R_{nl}(r)$ satisfying the following differential equation for NCPLHRT in the symmetries of (NC: 3D- RSP):

$$
\frac{d^2R_{nl}(r)}{dr^2} + 2 \left[ E_{nl} - \frac{Z}{2r} + \frac{\lambda}{2r^2} - \frac{2(l+1)}{2r^2} \right] R_{nl}(r) = 0
$$

(26)

The two terms which composed the expression of $H_{nl}(r)$ are proportional with two infinitesimals parameters ($\Theta$ and $\overline{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (21) by applying standard perturbation theory for their exact solutions at first order of two parameters $\Theta$ and $\overline{\theta}$.

### 3.2 The Exact Modified Spin-Orbital Spectrum of Hydrogenic Atoms Under NCPLHRT Interactions and Spontaneous Symmetry

To evaluate the above four factors, which depended with the parameters of potential ($\lambda, \varepsilon, Z$), we apply the following special integration, the results of eq. (29) (see below) depended with the generalized hypergeometric function $\,_{q}F_{p} (-m, \varepsilon, \varepsilon - \beta, -n + \varepsilon, \lambda + 1; 1)$ which obtained from the generalized function $\,_{q}F_{p} (\alpha_{1}, \ldots, \alpha_{q}, \beta_{1}, \ldots, \beta_{p}, z)$ for $p - 3$ and $q - 2$, in addition to the usual Gamma function $\Gamma(x)$. After straightforward calculations, we can obtain the explicitly results [23]:

$$
\begin{align*}
T_{1}(n, \lambda, g, Z) &= \gamma^{2} \int_{0}^{r} \frac{ln\Gamma(l+3/2)}{ln\Gamma(l+n+3/2)} k_{l} \Theta T_{1} \left( n, \lambda, g, Z \right) d \theta \\
T_{2}(n, \lambda, g, Z) &= \gamma^{2} \int_{0}^{r} \frac{ln\Gamma(l+3/2)}{ln\Gamma(l+n+3/2)} k_{l} \Theta T_{2} \left( n, \lambda, g, Z \right) d \theta \\
T_{3}(n, \lambda, g, Z) &= \lambda^{2} \int_{0}^{r} \frac{ln\Gamma(l+3/2)}{ln\Gamma(l+n+3/2)} k_{l} \Theta T_{3} \left( n, \lambda, g, Z \right) d \theta \\
&= \lambda^{2} \int_{0}^{r} \frac{ln\Gamma(l+3/2)}{ln\Gamma(l+n+3/2)} k_{l} \Theta T_{3} \left( n, \lambda, g, Z \right) d \theta
\end{align*}
$$

(28)

Thus, it is easily to obtain the explicitly results:
\[ T_{1}(n, \lambda, g, Z) = \frac{Z}{2} \lambda^{2}\Gamma(n - l + 3/2)\Gamma(n + l + 3/2) \sum F_{2}(-n, 2l, l - 1/2; -n + 2l, l + 3/2; 1) \]

\[ T_{2}(n, \lambda, g, Z) = \frac{\gamma}{2} \lambda^{2}\Gamma(n - l + 3/2)\Gamma(n + l + 3/2) \sum F_{2}(-n, 2l + 1, l + 3/2; -n + 2l + 1, l + 3/2; 1) \]

\[ T_{3}(n, \lambda, g, Z) = -\chi T_{1}(n, \lambda, g, Z) \]

\[ = \lambda \frac{\gamma}{n^{2}} \Gamma(n - l - 3/2)\Gamma(n + l + 3/2) \sum F_{2}(-n, 2l + 3l + 5/2; -n + 2l + 3l + 3/2; 1) \]

It is well known that \(I = [0, 1, ..., n - 1] \geq 0\), then \((-l - 1/2)\) and \((-l - 3/2)\) are representing two negative values and we having \(\Gamma(-l - 1/2) = \Gamma(-l - 3/2) = +\infty\), which gives

\[ T_{2}(n, \lambda, g, Z) = T_{1}(n, \lambda, g, Z) = T_{3}(n, \lambda, g, Z) = 0 \], allowed

\[ E_{\text{m-cb}}(n, \lambda, g, Z) = -\chi N^{2}\Theta \sum \frac{\Gamma(I + 3/2)\Gamma(n - l + 3/2)}{\Gamma(3/2 - l)\Gamma(I + l + 3/2)} k_{l} \lambda^{2} F_{2}(-n, 2l, l - 1/2; -n + 2l, l + 3/2; 1) \]

\[ E_{\text{d-cb}}(n, \lambda, g, Z) = -\chi N^{2}\Theta \sum \frac{\Gamma(I + 3/2)\Gamma(n - l + 3/2)}{\Gamma(3/2 - l)\Gamma(I + l + 3/2)} k_{l} \lambda^{2} F_{2}(-n, 2l, l - 1/2; -n + 2l, l + 3/2; 1) \]

Thus, our research also conveys another innovative feature exact, the extended global quantum group symmetry (NC: 3D-RSP) is broken simultaneously and replaced by new quantum subgroup symmetry (NC: 3D-RS) under interaction of hydrogenic atoms with NCPLHRT.

### 3.3 The Exact Modified Magnetic Spectrum of Hydrogenic Atoms Under NCPLHRT Interactions in Residual Group (NC: 3D-RS):

Further to the important previously obtained results, now, we consider another physically meaningful phenomena produced by the effect of NCPLHRT related to the influence of an external uniform magnetic field \(\vec{B}\), to avoid the repetition in the theoretical calculations, it’s sufficient to apply the following replacements:

\[ \Theta \to \chi \vec{B} \Rightarrow -\left( \frac{Z}{2r^{3}} + \frac{\lambda}{2r} + \frac{\chi}{2} \right) \Theta \to -\left( \frac{Z}{2r^{3}} + \frac{\lambda}{2r} + \frac{\chi}{2} \right) \vec{B} \vec{L} \]

\[ \frac{\partial}{\partial \vartheta} \to \frac{\partial}{\partial \vartheta} \vec{B} = \frac{\vartheta}{\partial \vartheta} \vec{B} \]

\[ E_{\text{mag-cb}}(n, m, \lambda, g, Z) = -\chi N^{2}\Theta \sum \frac{\Gamma(I + 3/2)\Gamma(n - l + 3/2)}{\Gamma(3/2 - l)\Gamma(I + l + 3/2)} \lambda^{2} F_{2}(-n, 2l, l - 1/2; -n + 2l, l + 3/2; 1) B_{m} \]

We have \(-l \leq m \leq l\), which allow us to fixing \((2l + 1)\) values for discreet number \(m\).

### 4. RESULTS AND DISCUSSION

We are now in a position to attack the main objective of our study, let us resume the nonrelativistic modified eigenenergies \(E_{\text{nc-wcb}}(n, j, l, s, m, \alpha)\) and \(E_{\text{nc-dcb}}(n, j, l, s, m, \alpha)\) of a hydrogenic atom under NCPLHRT obtained from solving MSE for \(n^{th}\) excited states in (NC: 3D-RSP) symmetries. On based to our original results presented on the Eqs. (31) and (34), in addition to the ordinary energy \(E_{\text{m}}\) for NCPLHRT, which presented in the eq. (8):
\[ E_{nc-ucb}(n, \lambda, g, Z, j, l, s, m) = \sqrt{2L}(2n + l + 3/2) + \sum_{j=0}^{n} \left(b^{(0)}_{j} \lambda^{j+1/2} + b^{(1)}_{j} \lambda^{j+1/4} + \lambda^{j} \right) \]

\[ -\gamma N^{\Theta} Z \left( \frac{\Gamma(1 + 3/2) \Gamma(n - 1 + 3/2)}{\Gamma(3/2 - l) \Gamma(l + n + 3/2)} \right) \lambda^{2l} F_{2}^{\Theta}(-n, 2l, l - 1/2, n) \]

and

\[ E_{nc-dcl}(n, \lambda, g, Z, j, l, s, m) = \sqrt{2L}(2n + l + 3/2) + \sum_{j=0}^{n} \left(b^{(0)}_{j} \lambda^{j+1/2} + b^{(1)}_{j} \lambda^{j+1/4} + \lambda^{j} \right) \]

\[ -\gamma N^{\Theta} Z \left( \frac{\Gamma(1 + 3/2) \Gamma(n - 1 + 3/2)}{\Gamma(3/2 - l) \Gamma(l + n + 3/2)} \right) \lambda^{2l} F_{2}^{\Theta}(-n, 2l, l - 1/2, n) \]

This is the main goal of this work. It’s clearly, that the obtained eigenvalues of energies are real’s and then the NC diagonal Hamiltonian \( H_{nc-dc} \) is Hermitian, furthermore it’s possible to writing the three elements: \( (H_{nc-dc})_{11} \), \( (H_{nc-dc})_{22} \) and \( (H_{nc-dc})_{33} \) of NC nonrelativistic Hamiltonian describing hydrogenic atoms with NCPLHRT as follows:

\[ \Delta_{nc} = \Delta - \frac{L \cdot \sigma \cdot \sigma}{2} \]

\[ H_{int-ucb} = -Z / r + g r + \lambda r^{2} - \gamma (k \Theta + \lambda^{N_{mod-3}}) \left( \frac{Z}{2r^{3}} + \frac{g^{2}}{2r} + \lambda \right) \]

\[ H_{int-dcl} = -Z / r + g r + \lambda r^{2} - \gamma (k \Theta + \lambda^{N_{mod-3}}) \left( \frac{Z}{2r^{3}} + \frac{g^{2}}{2r} + \lambda \right) \]

Thus, the ordinary kinetic term for CPLHRT \( \left(-\frac{\Delta}{2}\right) \) and ordinary interaction \( (-Z / r + g r + \lambda r^{2}) \) are replaced by new modified form of kinetic term \( \left(-\frac{\Delta_{nc}}{2}\right) \) and new modified interactions \( (H_{int-ucb} \text{ and } H_{int-dcl}) \). On the other hand, it is evident to consider the quantum number \( m \) takes \( (2l+1) \) values and we have also two values for \( j = l \pm 1/2 \), thus every state in usually three-dimensional space of energy for new NCPLHRT will be \( 2(2l+1) \) sub-states. To obtain the total complete degeneracy of energy level of NCPLHRT in NC 3-dimension space-phases, we need to sum for all allowed values of \( l \). Total degeneracy is thus,

\[ 2 \sum_{l=0}^{n} (2l + 1) = 2n^{2} \]

Note that the obtained new energy eigenvalues \( (E_{nc-ucb}(n, \lambda, g, Z, j, l, s, m)) \) and \( (E_{nc-dcl}(n, \lambda, g, Z, j, l, s, m)) \) depend to new discrete atomic quantum numbers \( (n, j, l, s) \) and \( m \) in addition to the parameters \( (\lambda, g, Z) \) of the NCPLHRT. Paying attention to the behaviour of the spectrums (35) and (36) \( (E_{nc-ucb}(n, \lambda, g, Z, j, l, s, m)) \) and \( (E_{nc-dcl}(n, \lambda, g, Z, j, l, s, m)) \), it is possible to recover the results of commutative space when we consider \((\Theta, Z) \rightarrow (0, 0)\).

5. CONCLUSION

In this paper three-dimensional MSE for NCPLHRT has been solved via Bopp’s shift method and independent time standard perturbation theory in (NC: 3D-RSP) symmetries, we resume the main obtained results:

1) The exact energy spectrum \( (E_{nc-ucb}(n, \lambda, g, Z, j, l, s, m)) \) and \( (E_{nc-dcl}(n, \lambda, g, Z, j, l, s, m)) \) for \( n^{th} \) excited levels, for hydrogenic atoms,
2) Ordinary interaction \( (-Z / r + g r + \lambda r^{2}) \) were replaced by \( NCPLHRT \) \((H_{int-ucb} \text{ and } H_{int-dcl})\) for hydrogenic atoms,
3) The ordinary kinetic term \(-\frac{\Delta}{2}\) modified to the new form \( \frac{\Delta_{nc}}{2} \) for NCPLHRT,
4) We have shown that, the group symmetry (NC: 3D-RSP) corresponding NCPLHRT were broken simultaneously and replaced by the new residual symmetry sub-group (NC: 3D-RS).
5) It has been shown that, the MSE presents useful
rich spectrums for improved understanding of hydrogenic atoms influenced by the NCPLHRT and we have seen also that the modified of spin-orbital and modified Zeeman effect were appears du the presence of the two infinitesimal parameters \((\Theta, \chi)\) which are induced by position-position noncommutativity property of space.

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