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New Nonrelativistic Quarkonium Masses in the Two-Dimensional Space-Phase using Bopp's shift Method and Standard Perturbation Theory

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In present work, the exact analytical bound-state solutions of modified Schrödinger equation (MSE) with modified extended Cornel potential (MECP) have been presented using both Bopp's shift method and standard perturbation theory in the noncommutative two dimensional real space and phase (NC-2D: RSP), we have also constructed the corresponding noncommutative Hamiltonian operator which containing two new terms, the first one is modified Zeeman effect and the second is spin-orbital interaction. The theoretical results show that the automatically appearance for both spin-orbital interaction and modified Zeeman effect leads to the degenerate to energy levels to $2(2l+1)$ sub states.

Keywords: Schrödinger equation, Star product, Bopp's shift method, Extended Cornel potential.

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1. INTRODUCTION

The concept of non-relativistic symmetries of the Schrödinger Hamiltonian discovered centuries ago were recognized empirically in many field of sciences for example spectroscopy, atoms, molecules, and nuclei by using numerous methods such as quasi-linearization method, Hill determinant method, point canonical transformation, numerical method, Nikiforov-Uvarov method, Laplace Transform method, SUSQM method, power series technique and the analytical exact iteration method. The non-relativistic Schrödinger equation which describes the motion of spin 1/2 particle has been successfully used in solving many physical problems in a lot of heavy quarkonium systems and low-energy physics [1-5]. Recently, the symmetries were extended to new space-phase known by noncommutative space and phase to obtain profound interpretation in Nano and plank's scales, much work in case of the noncommutative space-phase at two, three and N generalized dimensions has been done for solving the three fundamental equations [6-10] and in particularly, our previously works [11-18] . The notions of noncommutativity of space and phase developed on based to the Seiberg-Witten map, Bopp's shift method and the star product, defined on the first order of two infinitesimal parameters antisymmetric $2\left(\theta^{\mu\nu},\bar{\theta}^{\mu\nu}\right)\equiv\varepsilon^{ijk}\left(\theta_k,\bar{\theta}_k\right)$ as $(\hbar=c=1)$ [6-11]:

$$f(x)*g(x)=f(x)g(x)-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f(x)\partial_\nu^x g(x)-\\-\frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f(x)\partial_\nu^p g(x). \quad (1)$$

Thus, the noncommutativity commutators of the coordinates $\left[\hat{x}_i,\hat{x}_j\right]_*$ and corresponding momentums $\left[\hat{p}_i,\hat{p}_j\right]_*$ can be described by the following commutations relations:

$$\left[\hat{x}_i,\hat{x}_j\right]_*=i\theta_{ij}\\ \left[\hat{p}_i,\hat{p}_j\right]_*=i\bar{\theta}_{ij}. \quad (2)$$

The simplest case corresponds to θ_{ij} and $\bar{\theta}_{ij}$ being constants, which we call non-dynamical or $\left(\theta-\bar{\theta}\right)$ NC spaces-phases. It's important to notice, that the Bopp's shift method will be apply in this paper instead of solving the MSE with star product, the SE will be treated by using directly usual commutators on quantum mechanics, in addition to the following two commutators [10-14]:

$$\left[\hat{x}_i,\hat{x}_j\right]=i\theta_{ij} \text{ and } \left[\hat{p}_i,\hat{p}_j\right]=i\bar{\theta}_{ij}. \quad (3)$$

In this paper we are using noncommutative theories in (NC: 2D-RSP) model to find out what will happen for 2D nonrelativistic spectrum if effects of noncommutativity of both space and phase are considered for MECP that governs the new extended Cornell potential $V_{nc-mec}(\hat{r})$:

$$V_{nc-mec}(\hat{r})=V(r)+\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)\theta L_z+\frac{\bar{\theta}L_z}{2\mu}. \quad (4)$$

On based of two main references [4] and [14], to discover the new spectrum of energy and a possibility of obtain new applications in different fields ($V(r)$ is given in the next section). The rest of present search is organized as follows: In next section, we give briefly review to the SE with ECP in 2D spaces. In section 3, we shall briefly introduce the fundamental concepts of Bopp's shift method and then we apply this method to derive the MECP and the corresponding noncommutative (spin-orbital and new Zeeman) Hamiltonians operators and the corresponding two spectrums by applying perturbation theory for ground stat and first excited states. In section 4, we conclude the global noncommutative Hamiltonian and we resume the global spectrum

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for MECP in first order of two infinitesimal parameters' θ and $\bar{\theta}$ in (NC-2D: RSP) symmetries. Next, we calculate the mass spectra of heavy quarkonia in the 2*D* space-phase. Finally, the important found results and the conclusions are discussed in the last section.

2. REVIEW OF THE EIGENVALUES AND EIGNENFUNCTIONS FOR MECP IN 2D

In this section, we shall review the eigenvalues and eignenfunctions for spherically symmetric for the potential known by ECP in 2D spaces [4-5]:

$$V(r) = ar^2 + br - \frac{c}{r} - \frac{d}{r^2}. \tag{5}$$

The four parameters: a , b , c and d are constants, the above confining interaction potential consisting of a sum of harmonic, linear, Coulombic and pseudoharmonic potential terms, the last term is incorporated into the quarkonium potential for the sake of coherence while the rest terms represents the Cornell potential ,the complex eignenfunctions $\Psi(r,\varphi)$ in 2D space for above potential satisfied the SE in spherical coordinates is [4]:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{l^2}{r^2} + 2\mu\left(E_{nl} - ar^2 - br + \frac{c}{r} + \frac{d}{r^2}\right)\right)\Psi(r,\phi) = 0, \tag{6}$$

where μ and l are the reduced mass of the two particles and the angular quantum number, respectively while $E_{n,l}$ is the total energy of the particle. Now, inserting the new form of $\Psi(r,\varphi)$ [4]:

$$\Psi(r,\phi) = \frac{1}{r^{1/2}}\varphi_n(r)\exp(\pm im\phi). \tag{7}$$

Eq. (6) reduces to [4]:

$$\begin{aligned}&\frac{d^2\varphi_n(r)}{dr^2} + 2\mu\Big[\varepsilon_{nl} - a_1 r^2 - b_1 r + \\ &+\frac{c_1}{r} + \frac{d_1 + 1/4 - l^2}{r^2}\Big]\varphi_n(r) = 0.\end{aligned} \tag{8}$$

With the simplifications $(\varepsilon_{nl}, a_1, b_1, c_1, d_1) = 2\mu(E_{nl}, a, b, c, d)$ and then, the complete normalized wave functions and corresponding energies for n^{th} excited state in 2D space, respectively [4]:

$$\begin{aligned}&\Psi^{(l)}(\vec{r}) = N_n \prod_{i=1}^{n}\left(r - \alpha_i^{(n)}\right) r^{l'/2} \times \\ &\times \exp\left(-\tfrac{1}{2}\sqrt{2\mu a}r^2 - \sqrt{\frac{\mu}{2a}}br\right)\exp(\pm im\phi) \\ &E_{nl} = \sqrt{\frac{a}{2\mu}}(2 + 2n + l') - \frac{b^2}{4a},\end{aligned} \tag{9}$$

where $l' = \sqrt{4l^2 - 8\mu d}$ and (N_0, N_1) are two normalized constants.

3. NONCOMMUTATIVE PHASE-SPACE HAMILTONIAN OPERATOR FOR MECP

3.1 Formalism of Bopp's Shift

In this sub-section, on based to our previously works [15-18], we give a brief review to the fundamental principles of modified Schrödinger equation in (NC-2*D*: RSP), to achieve this goal we apply the important 4-steps on the ordinary SE:

- we replace ordinary 2*D* Hamiltonian operator $\hat{H}(p_i, x_i)$ by noncommutative new Hamiltonian operator $\hat{H}(\hat{p}_i, \hat{x}_i)$,

- we replace ordinary complex wave function $\Psi(\vec{r})$ by new complex wave function $\hat{\Psi}(\vec{\hat{r}})$,

- we replace ordinary energy $E_{n,l}$ by noncommutative energy E_{nc} ,

- the forth step correspond to replace the ordinary old product by new star product.

Which allow us to constructing the modified SE in (NC-2D: RSP) symmetries as:

$$\hat{H}(\hat{p}_i, \hat{x}_i) * \hat{\Psi}(\vec{\hat{r}}) = E_{nc}\hat{\Psi}(\vec{\hat{r}}). \tag{10}$$

The Bopp's shift method allows finding the reduced following SE without star product:

$$H(\hat{p}_i, \hat{x}_i)\psi(\vec{r}) = E_{nc}\psi(\vec{r}), \tag{11}$$

the modified Hamiltonian $H(\hat{p}_i, \hat{x}_i)$ defined as a function of two operators $\hat{x}_i$ and $\hat{p}_i$:

$$H_{nc-mecp}(\hat{p}_i, \hat{x}_i) = \frac{\hat{p}_i^{\,2}}{2\mu} + V_{nc}(\hat{r}), \tag{12}$$

the modified 2D potential $V_{nc-mecp}(\hat{r})$ obtained by the following procedure:

$$V_{nc-mecp}(\hat{r}) = a\hat{r}^2 + b\hat{r} - \frac{c}{\hat{r}} - \frac{d}{\hat{r}^2}. \tag{13}$$

the two operators $\hat{x}_i$ and $\hat{p}_i$ in (NC-2D: RSP) symmetries are given by [14-18]:

$$\hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j \quad \text{and} \quad \hat{p}_i = p_i + \frac{\bar{\theta}_{ij}}{2}x_j. \tag{14}$$

On based to our references [15-18], we can write the two operators $\hat{r}^2$ and $\hat{p}^2$ in NC 2D spaces and phases as follows:

$$\hat{r}^2 = r^2 - \theta L_z \quad \text{and} \quad \frac{\hat{p}^2}{2\mu} = \frac{p^2}{2\mu} + \frac{\bar{\theta}L_z}{2\mu}. \tag{15}$$

After straightforward calculations one can obtains the different terms in (NC-2*D*: RSP) as follows:

$$a\hat{r}^2 = ar^2 - a\theta L_z, \qquad b\hat{r} = br - \frac{b}{2r}\theta L_z$$
$$\frac{c}{\hat{r}} = \frac{c}{r} + \frac{c}{2r^3}\theta L_z \quad \text{and} \quad \frac{d}{\hat{r}^2} = \frac{d}{r^2} + \frac{d}{r^4}\theta L_z. \qquad (16)$$

Which allow us to writing the modified 2D potential $V_{nc-mecp}(\hat{r})$ in (NC-2D: RSP) symmetries as follows:

$$V_{nc-mecp}(\hat{r}) = ar^2 + br - \frac{c}{r} - \frac{d}{r^2} + V_{pert}(r,\theta,\bar{\theta}). \qquad (17)$$

It's clearly that, the first 4-terms in eq. (17) represent the ordinary ECP while the rest term $V_{pert}(r,\theta,\bar{\theta})$ is produced by the deformation of space and phase. The global perturbative potential operators $V_{pert-mcep}(r,\theta,\bar{\theta})$ for studied potential MECP in (NC-2D: RSP) symmetries will be written as:

$$V_{pert-mecp}(r,\theta,\bar{\theta}) = \left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right)\theta L_z + \frac{\bar{\theta}L_z}{2\mu}. \qquad (18)$$

3.2 The Spin-orbital Noncommutative Hamiltonian for MECP in (NC: 2*D*- RSP)

In order, to discover the new contribution of MECP, we replace the two couplings θL_z and $\bar{\theta} L_z$ by $\alpha\Theta\vec{S}\vec{L}$ and $\alpha\bar{\theta}\overleftrightarrow{S}\overleftrightarrow{L}$, respectively, then the above perturbed operator becomes as:

$$H_{\text{pert-mec}}(r,\Theta,\bar{\theta}) = \alpha\left\{\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{\mu}\right\}\mathbf{L}\vec{S}. \qquad (19)$$

Here $\overleftrightarrow{S}$ denote the spin of a quarkonium system. Now, we replace the spin-orbital interaction $\mathbf{L}\vec{S}$ by $G^2 = \frac{1}{2}\left(\overleftrightarrow{J}^2 - \overleftrightarrow{L}^2 - \overleftrightarrow{S}^2\right)$ to obtain the new physical form:

$$H_{\text{pert-mcph}}(r,\Theta,\bar{\theta}) = \left\{\alpha\theta\left(\frac{3c}{r^8} + \frac{2b}{r^6} - a\right) + \frac{\bar{\theta}}{2\mu}\right\}G^2. \qquad (20)$$

As it well known, ($\overleftrightarrow{J}^2$, $\overleftrightarrow{L}^2$, $\overleftrightarrow{S}^2$ and s_z) formed complete basis on quantum mechanics, then the operator $\left(\overleftrightarrow{J}^2 - \overleftrightarrow{L}^2 - \overleftrightarrow{S}^2\right)$ will be gives 2-eigenvalues $k_{\mp} \equiv \frac{1}{2}\left\{\left(l \pm \frac{1}{2}\right)\left(l + \frac{1}{2} \pm 1\right) + l(l+1)(l+1) - \frac{3}{4}\right\}$, corresponding $j = l \pm \frac{1}{2}$ respectively (for $\vec{s} = \overrightarrow{1/2}$). Then, one can form a diagonal matrix $\hat{H}_{so-mecp}$ of order (2×2), with non null elements $\left(\hat{H}_{so-mecp}\right)_{11}$ and $\left(\hat{H}_{so-mecp}\right)_{22}$ for MECP in (NC-2D: RSP) symmetries:

$$\left(\hat{H}_{so-mecp}\right)_{11} = k_+\alpha\left\{\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right\}$$
for $j = l + \frac{1}{2} \Rightarrow$ spin up
$$\left(\hat{H}_{so-mecp}\right)_{22} = k_-\alpha\left\{\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right\} \qquad (21)$$
for $j = l - \frac{1}{2} \Rightarrow$ spin down.

After profound straightforward calculation, one can show that, the radial function $\phi_n(r)$ satisfied the following differential equation for MECP in (NC-2D: RSP) symmetries:

$$\frac{d^2\varphi_n(r)}{dr^2} + 2\mu\left[\varepsilon_{nl} - a_1r^2 - b_1r + \frac{c_1}{r} + \frac{d_1 + 1/4 - l^2}{r^2} - \right.$$
$$\left. -\alpha\left\{\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{\mu}\right\}\mathbf{L}\vec{S}\right]\varphi_n(r) = 0. \qquad (22)$$

3.3 The Exact Spectrum of Ground States Produced by NC Spin-orbital Hamiltonian $\hat{H}_{so-mecp}$ for MECP in (NC: 2*D*- RSP) Symmetries

Now, the aim of this subsection is to obtain the modifications to the energy levels for ground states

$E_{\text{u-mecp}}(a,b,c,d)$ and $E_{\text{d-mecp}}(a,b,c,d)$

for spin up and spin down, respectively, at first order of two parameters θ and $\bar{\theta}$. In order to achieve this goal, we apply the standard perturbation theory:

$$\frac{E_{\text{u-mecp}}(a,b,c,d)}{\alpha 2\prod|N_0|^2} = k_+\int_0^{+\infty} r^{l'+1}\exp\left(-\beta r^2 - \gamma r\right)\left(\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right)dr. \qquad (23)$$

$$\frac{E_{\text{d-mecp}}(a,b,c,d)}{\alpha 2\prod|N_0|^2} = k_-\int_0^{+\infty} r^{l'+1}\exp\left(-\beta r^2 - \gamma r\right)\left(\theta\left(\frac{d}{r^4} + \frac{c}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right)dr. \qquad (24)$$

Where $\beta = \sqrt{2\mu a}$ and $\gamma = \frac{4\mu c}{l'+1}$, a direct simplification gives:

$$E_{\text{u-mecp}}(a,b,c,d) = 2\prod|N_0|^2 k_+\left\{\theta\left(\sum_{i=1}^{4}T_i\right) + \frac{\bar{\theta}}{2\mu}T_5\right\} \qquad (25)$$

$$E_{\text{d-mecp}}(a,b,c,d) = 2\prod|N_0|^2 k_-\left\{\theta\left(\sum_{i=1}^{4}T_i\right) + \frac{\bar{\theta}}{2\mu}T_5\right\}. \qquad (26)$$

Where, the five terms $T_i\left(i=\overline{1,5}\right)$ are given by:

$$\begin{aligned} T_1 &= d\int_0^{+\infty} r^{(l'-2)-1}\exp\left(-\beta r^2-\gamma r\right)dr, \\ T_2 &= \frac{c}{2}\int_0^{+\infty} r^{(l'-1)-1}\exp\left(-\beta r^2-\gamma r\right)dr, \\ T_3 &= -\frac{b}{2}\int_0^{+\infty} r^{(l'+1)-1}\exp\left(-\beta r^2-\gamma r\right)dr, \\ T_4 &= -aT_5 = -a\int_0^{+\infty} r^{(l'+2)-1}\exp\left(-\beta r^2-\gamma r\right)dr. \end{aligned} \quad (27)$$

In order to obtain the above integrals, we applying the following special integration [19]:

$$\int_0^{+\infty} x^{\nu-1}\exp\left(-\beta x^2-\gamma x\right)dx = (2\beta)^{-\frac{\nu}{2}}\Gamma(\nu)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right). \quad (28)$$

Where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$ denote to the Parabolic cylinder functions function, $\Gamma(\nu)$ Gamma function $\mathrm{Re}l(\beta)\rangle 0$ and $\mathrm{Re}l(\nu\rangle 0)$. After straightforward calculations, we can obtain the explicitly results:

$$\begin{aligned} T_1 &= d(2\beta)^{-\frac{l'-2}{2}}\Gamma(l'-2)D_{-(l'-2)}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \\ T_2 &= \frac{c}{2}(2\beta)^{-\frac{l'-1}{2}}\Gamma(l'-1)D_{-(l'-1)}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \\ T_3 &= -\frac{b}{2}(2\beta)^{-\frac{l'+1}{2}}\Gamma(l'+1)D_{-(l'+1)}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \\ T_4 &= -aT_5 = -a(2\beta)^{-\frac{l'+2}{2}}\Gamma(l'+2)D_{-(l'+2)}\left(\frac{\gamma}{\sqrt{2\beta}}\right). \end{aligned} \quad (29)$$

Inserting the above expressions into equations (22) and (23), one obtains the following results for exact modifications of ground states $E_{\text{u-mecp}}(a,b,c,d)$ and $E_{\text{d-mecp}}(a,b,c,d)$ produced by new spin-orbital effect for MECP:

$$\begin{aligned} E_{\text{u-mecp}}(a,b,c,d) &= 2\Pi|N_0|^2 k_+\left\{\theta T_{nc-0s}+\bar{\theta}T_{nc-0p}\right\} \\ E_{\text{d-mecp}}(a,b,c,d) &= 2\Pi|N_0|^2 k_-\left\{\theta T_{nc-0s}+\bar{\theta}T_{nc-0p}\right\}. \end{aligned} \quad (30)$$

Where the two factors T_{nc-0} and T_{nc-0p} are given by:

$$T_{nc-0s} \equiv \sum_{i=1}^{4} T_i \quad \text{and} \quad T_{nc-0pis} \equiv T_5. \quad (31)$$

It's important to notice that the above two terms T_{nc-0s} and T_{nc-0p} are represent the noncommutative geometry of space and phase, respectively.

3.4 The Exact Spectrum of First Excited States Produced by Noncommutative Spin-orbital Hamiltonian $\hat{H}_{so-mecp}$ for MECP in (NC: 3D-RSP)

The aim of this subsection is to obtain the new modifications to the energy levels for first excited states $E_{\text{u--mecp1}}(n=1,a,b,c,d)$ and $E_{\text{d--mecp1}}(n=1,a,b,c,d)$ corresponding spin up and spin down, respectively at first order of two parameters θ and $\bar{\theta}$ for MECP which are obtained by applying the standard perturbation theory as:

$$\frac{E_{\text{u--mecp1}}(n=1,a,b,c,d)}{2\Pi|N_1|^2} = k_+\int_0^{+\infty}\left(r^{l'+3}+\left(\alpha_1^{(1)}\right)^2 r^{l'+1}-2\alpha_1^{(1)}r^{l'+2}\right)\exp\left(-\alpha' r^2-\beta' br\right)\begin{pmatrix}\Theta\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)\\ +\frac{\bar{\theta}}{2\mu}\end{pmatrix}dr \quad (32)$$

$$\frac{E_{\text{d--mecp1}}(n=1,a,b,c,d)}{2\Pi|N_1|^2} = k_-\int_0^{+\infty}\left(r^{l'+3}+\left(\alpha_1^{(1)}\right)^2 r^{l'+1}-2\alpha_1^{(1)}r^{l'+2}\right)\exp\left(-\alpha' r^2-\beta' br\right)\begin{pmatrix}\Theta\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)\\ +\frac{\bar{\theta}}{2\mu}\end{pmatrix}dr \quad (33)$$

Where (α', β') are equals $(\sqrt{2\mu a}, 2\sqrt{\frac{\mu}{2a}}b)$and then a direct simplification to the above equations (31) and (32) gives:

$$\begin{aligned} &E_{\text{u--mecp1}}(n=1,a,b,c,d) = \\ &= 2\Pi|N_1|^2 k_+\left\{\Theta\sum_{i=1}^{12}L_i+\frac{\bar{\theta}}{2m_0}\sum_{i=13}^{15}T_i\right\} \end{aligned} \quad (34)$$

$$\begin{aligned} &E_{\text{d--mecp1}}(n=1,a,b,c,d) = \\ &= 2\Pi|N_1|^2 k_-\left\{\Theta\sum_{i=1}^{12}L_i+\frac{\bar{\theta}}{2m_0}\sum_{i=13}^{15}T_i\right\}. \end{aligned} \quad (35)$$

Where, the 15- terms $L_i\left(i=\overline{1,15}\right)$ are given by:

$$
\begin{aligned}
L_1 &= d\int_0^{+\infty} r^{l'-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_2 &= \frac{c}{2}\int_0^{+\infty} r^{(l'+1)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_3 &= -\frac{b}{2}\int_0^{+\infty} r^{(l'+3)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_4 &= -a\int_0^{+\infty} r^{(l'+4)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
L_5 &= \left(\alpha_1^{(1)}\right)^2 d\int_0^{+\infty} r^{(l'-2)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_6 &= \left(\alpha_1^{(1)}\right)^2 \frac{c}{2}\int_0^{+\infty} r^{(l'-1)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_7 &= -\left(\alpha_1^{(1)}\right)^2 \frac{b}{2}\int_0^{+\infty} r^{(l'+1)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_8 &= -a\left(\alpha_1^{(1)}\right)^2\int_0^{+\infty} r^{(l'+2)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
L_9 &= -2\alpha_1^{(1)}d\int_0^{+\infty} r^{(l'-1)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_{10} &= -\alpha_1^{(1)}c\int_0^{+\infty} r^{l'-1}\exp\left(-\alpha' r^2-\beta' br\right)dr\\
L_{11} &= \alpha_1^{(1)}b\int_0^{+\infty} r^{(l'+2)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_{12} &= 2a\alpha_1^{(1)}\int_0^{+\infty} r^{(l'+3)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
L_{13} &= \int_0^{+\infty} r^{l'+4-1}\exp\left(-\alpha' r^2-\beta' br\right)dr,\\
L_{14} &= \left(\alpha_1^{(1)}\right)^2\int_0^{+\infty} r^{(l'+2)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr\\
L_{15} &= -2\alpha_1^{(1)}\int_0^{+\infty} r^{(l'+3)-1}\exp\left(-\alpha' r^2-\beta' br\right)dr
\end{aligned}
\tag{39}
$$

In order to obtain the results of above equations, we apply the special integral which represents by eq. (28):

$$
\begin{aligned}
L_1 &= d\left(2\beta'\right)^{-\frac{l'}{2}}\Gamma\left(l'\right)D_{-l'}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right),\\
L_2 &= \frac{c}{2}\left(2\beta'\right)^{-\frac{l'+1}{2}}\Gamma\left(l'+1\right)D_{-(l'+)1}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right)\\
L_3 &= -\frac{b}{2}\left(2\beta'\right)^{-\frac{'+3}{2}}\Gamma\left(l'+3\right)D_{-(l'+)3}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right),\\
L_4 &= -a\left(2\beta'\right)^{-\frac{l'+4}{2}}\Gamma\left(l'+4\right)D_{-(l'+4)}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right)
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
L_5 &= \left(\alpha_1^{(1)}\right)^2 d\left(2\beta'\right)^{-\frac{l'-2}{2}}\Gamma\left(l'-2\right)D_{-(l'-2)}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right),\\
L_6 &= \left(\alpha_1^{(1)}\right)^2 \frac{c}{2}\left(2\beta\right)^{-\frac{\nu}{2}l'-1}\Gamma\left(l'-1\right)D_{-(l'-)1}\left(\frac{\gamma}{\sqrt{2\beta}}\right),\\
L_7 &= -\left(\alpha_1^{(1)}\right)^2 \frac{b}{2}\left(2\beta'\right)^{-\frac{l'+1}{2}}\Gamma\left(l'+1\right)D_{-(l'+)1}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right),\\
L_8 &= -a\left(\alpha_1^{(1)}\right)^2\left(2\beta'\right)^{-\frac{l'+2}{2}}\Gamma\left(l'+2\right)D_{-(l'+2)}\left(\frac{\gamma'}{\sqrt{2\beta'}}\right)
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
L_9 &= -2\alpha_1^{(1)}d\left(2\beta'\right)^{-\frac{l'-1}{2}}\Gamma\left(l'-1\right)D_{-(l'-1)}\left(\frac{\gamma'}{\sqrt{2\beta}}\right),\\
L_{10} &= -\alpha_1^{(1)}c\left(2\beta\right)^{-\frac{l'}{2}}\Gamma\left(l'\right)D_{-l'}\left(\frac{\gamma'}{\sqrt{2\beta}}\right),\\
L_{11} &= \alpha_1^{(1)}b\left(2\beta\right)^{-\frac{l'+2}{2}}\Gamma\left(l'+2\right)D_{-(l'+2)}\left(\frac{\gamma'}{\sqrt{2\beta}}\right),\\
L_{12} &= 2a\alpha_1^{(1)}\left(2\beta\right)^{-\frac{l'+3}{2}}\Gamma\left(l'+3\right)D_{-(l'+)3}\left(\frac{\gamma'}{\sqrt{2\beta}}\right)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
L_{13} &= \left(2\beta\right)^{-\frac{l'+4}{2}}\Gamma\left(l'+4\right)D_{-(l'+4)}\left(\frac{\gamma}{\sqrt{2\beta}}\right),\\
L_{14} &= \left(\alpha_1^{(1)}\right)^2\left(2\beta\right)^{-\frac{l'+2}{2}}\Gamma\left(l'+2\right)D_{-(l'+2)}\left(\frac{\gamma'}{\sqrt{2\beta}}\right)\\
L_{15} &= -2\alpha_1^{(1)}\left(2\beta\right)^{-\frac{l'+3}{2}}\Gamma\left(l'+2\right)D_{-(l'+3)}\left(\frac{\gamma'}{\sqrt{2\beta}}\right).
\end{aligned}
\tag{43}
$$

The above obtained explicitly results allow us to getting the exact modifications

$$E_{\text{u--mecp1}}\left(n=1,a,b,c,d\right)$$

and

$$E_{\text{d--mecp1}}\left(n=1,a,b,c,d\right)$$

of degenerated first excited states corresponding two polarized states produced by new spin-orbital Hamiltonian operator $\hat{H}_{so-mecp}$:

$$
\begin{aligned}
&E_{\text{u--mecp1}}\left(n=1,a,b,c,d\right)=\\
&=2\prod\left|N_1\right|^2 k_+\left\{\theta L_{nc-1s}+\frac{\bar{\theta}}{2\mu}L_{nc-1p}\right\}
\end{aligned}
\tag{44}
$$

and

$$
\begin{aligned}
&E_{\text{d--mecp1}}\left(n=1,a,b,c,d\right)=\\
&=2\prod\left|N_1\right|^2 k_-\left\{\theta L_{nc-1s}+\frac{\bar{\theta}}{2\mu}L_{nc-1p}\right\}
\end{aligned}
\tag{45}
$$

Where the two factors L_{nc-1} and L_{nc-1p} are given by the following form, respectively:

$$L_{nc-1s}\equiv\sum_{i=1}^{12}L_i \qquad \text{and} \qquad L_{nc-1p}\equiv\sum_{i=13}^{15}L_i \tag{46}$$

3.5 The Exact Spectrum Produced by Noncommutative Magnetic Hamiltonian $\hat{H}_{m-mecp}$ for MECP in (NC: 2D- RSP) Symmetries

On other hand, it's possible to found another automatically symmetry for MECP related to the influence of an external uniform magnetic field, generated from the effect of the new geometry of space and phase, it's deduced by the two following two replacements:

$$\Theta\to\chi B \qquad \text{and} \qquad \bar{\theta}\to\bar{\sigma}B \tag{47}$$

Here χ and $\bar{\sigma}$ are infinitesimal real two proportional's constants and to simplified the calculations we choose the magnetic field $\overleftrightarrow{B} = B\overleftrightarrow{k}$ and then we can make the following translation:

$$\left\{\theta\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\theta}}{2\mu}\right\}\vec{B}\vec{L}\to \to\left(\chi\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\sigma}}{2\mu}\right) \tag{48}$$

Which allow us to introduce the modified new magnetic Hamiltonian $\hat{H}_{m-mecp}$ in (NC-2D: RSP) for MECP as:

$$\hat{H}_{m-mecp}=\left(\chi\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\sigma}}{2\mu}\right)\left(\vec{B}\vec{J}+\hat{H}_z\right) \tag{49}$$

Here $\hat{H}_z \equiv -\vec{S}\vec{B}$ denote to the ordinary operator of Hamiltonian for of Zeeman Effect in quantum mechanics. To obtain the exact noncommutative magnetic modifications of energy ($E_{\text{mag-0}}(n=0,m,a,b,c,d)$, $E_{\text{mag-1}}(n=1,m,a,b,c,d)$) for MECP we just replace the 3-parameters k_+, Θ and $\bar{\theta}$ in the Eqs. (30) and (34) by the following new parameters m $(-l \le m \le +l)$, χ and $\bar{\sigma}$, respectively:

$$E_{\text{mag-0}}(n=0,m,a,b,c,d)= =2\prod|N_0|^2 Bm\left(\chi T_{nc-0s}+\frac{\bar{\sigma}T_{nc-0p}}{2\mu}\right) \tag{50}$$

$$E_{\text{mag-1}}(n=1,m,a,b,c,d)= =2\prod|N_1|^2 Bm\left(\chi T_{nc-1s}+\frac{\bar{\sigma}T_{nc-1p}}{2\mu}\right). \tag{51}$$

Where $E_{\text{mag-0}}(n=0,m,a,b,c,d)$ and $E_{\text{mag-1}}(n=1,m,a,b,c,d)$ are the exact magnetic modifications of spectrum corresponding the ground states and first excited states, respectively. It should be noted that coefficients $T_i\left(i=\overline{1,4}\right)$ and $L_i\left(i=\overline{1,15}\right)$ are the same as those found in our reference [14].

4. THE NEW GLOBAL EXACT SPECTRUM OF LOWEST EXCITED STATES FOR MECP IN (NC-2D: RSP) PRODUCED BY THE DIAGONAL ELEMENTS OF NONCOMMUTATIVE HAMILTONIAN OPERATOR $\hat{H}_{nc-mecp}$

It's clearly, that the obtained previous results which are presented by Eqs. (30), (44), (45), (50) and (51) of eigenvalues of energies are reels and then the noncommutative diagonal Hamiltonian operator $\hat{H}_{nc-mecp}$ will be Hermitian operator. Furthermore, we can obtain the explicit physical form of this operator on based to the results (21) and (49) for MECP, its represent by diagonal noncommutative matrix of order (2×2), with elements $\left(\hat{H}_{nc-mecp}\right)_{11}$ and $\left(\hat{H}_{nc-mecp}\right)_{22}$ in (NC-2D: RSP) symmetries:

$$\left(\hat{H}_{nc-mecp}\right)_{11}=-\frac{\Delta}{2\mu}+ar^2+br-\frac{c}{r}-\frac{d}{r^2}+k_+\left\{\theta\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\theta}}{2\mu}\right\}+\left(\chi\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\sigma}}{2\mu}\right)BL_z \tag{52}$$

if $j=l+\frac{1}{2}$ $\Rightarrow$ spin up

$$\left(\hat{H}_{nc-mecp}\right)_{22}=-\frac{\Delta}{2\mu}+ar^2+br-\frac{c}{r}-\frac{d}{r^2}+k_-\left\{\Theta\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\theta}}{2\mu}\right\}+\left(\chi\left(\frac{d}{r^4}+\frac{c}{2r^3}-\frac{b}{2r}-a\right)+\frac{\bar{\sigma}}{2\mu}\right)BL_z \tag{53}$$

if $j=l-\frac{1}{2}$ $\Rightarrow$ spin -down

Now we find the corresponding global modified energies ($E_{\text{nc u0}}(n,=0,l,m,s,a,b,c,d)$ - $E_{\text{nc d0}}(n=0,l,m,s,a,b,c,d)$) and ($E_{\text{nc u1}}(n=1,l,s,m,a,b,c,d)$ - $E_{\text{nc d1}}(n=1,l,s,m,a,b,c,d)$) for ground and first excited states of a particle fermionic with spin up and spin moving in the MECP, referring to Eqs. (30), (44), (45), (50) and (51), we find the results as follows:

$$E_{\text{nc u0}}(n,=0,l,m,s,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(2+l')-\frac{2\mu c^2}{(l'+1)^2}+2\prod|N_0|^2\left\{\begin{array}{l}k_+\left(\theta T_{nc-0s}+\frac{\bar{\theta}}{2\mu}T_{nc-0p}\right)\\+Bm\left(\chi T_{nc-0s}+\frac{\bar{\sigma}T_{nc-0p}}{2\mu}\right)\end{array}\right\} \tag{54}$$

$$E_{\text{nc d0}}(n=0,l,m,s,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(3+l')-\frac{2\mu c^2}{(l'+1)^2}+2\prod|N_0|^2\left\{\begin{array}{l}k_-\left(\theta T_{nc-0s}+\frac{\bar{\theta}}{2\mu}T_{nc-0p}\right)\\+Bm\left(\chi T_{nc-0s}+\frac{\bar{\sigma}T_{nc-0p}}{2\mu}\right)\end{array}\right\} \tag{55}$$

and

$$E_{\text{nc u1}}(n=1,l,s,m,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(4+l')-\frac{b^2}{4a}+2\prod|N_1|^2\begin{Bmatrix}k_+\left(\theta L_{nc-1s}+\bar{\theta}L_{nc-1p}\right)\\+Bm\left(\chi T_{nc-1s}+\frac{\bar{\sigma}T_{nc-1p}}{2\mu}\right)\end{Bmatrix} \quad (56)$$

and

$$E_{\text{nc d1}}(n=1,l,s,m,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(4+l')-\frac{b^2}{4a}+2\prod|N_1|^2\begin{Bmatrix}k_-\left(\theta L_{nc-1s}+\bar{\theta}L_{nc-1p}\right)\\+Bm\left(\chi T_{nc-1s}+\frac{\bar{\sigma}T_{nc-1p}}{2\mu}\right)\end{Bmatrix} \quad (57)$$

As it's mentioned in our previously works [14-18], the atomic quantum number m can be takes $(2l+1)$ values and we have also two values for $j=l\pm\frac{1}{2}$, thus every state in usually $2D$ space of MECP will be in (NC-2D: RSP): $2(2l+1)$ sub-states. As it well known, the eigenvalues j of the total operator $\vec{J}=\vec{L}+\vec{S}$ can be obtains from the interval $|l-s|\le j\le|l+s|$, which allow us to obtaining the eigenvalues $k(j,l,s)\equiv j(j+1)+l(l+1)-s(s+1)$ of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$, thus, by substituting $k_\mp$ by $k(j,l,s)$ into E_{qs}. (56) and (58) to obtain $E_{\text{nc mecp0}}(n=0,l,s,a,b,c,d)$ and $E_{\text{nc mecp1}}(n=1,l,s,a,b,c,d)$ for ground state and first excited states:

$$E_{\text{nc mecp0}}(n=0,l,s,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(2+l')-\frac{2\mu c^2}{(l'+1)^2}+2\prod|N_0|^2\begin{Bmatrix}k(j,l,s))\left(\theta T_{nc-0s}+\frac{\bar{\theta}}{2\mu}T_{nc-0p}\right)\\+Bm\left(\chi T_{nc-0s}+\frac{\bar{\sigma}T_{nc-0p}}{2\mu}\right)\end{Bmatrix} \quad (58)$$

$$E_{\text{nc mecp1}}(n=1,l,s,a,b,c,d)=\sqrt{\frac{a}{2\mu}}(4+l')-\frac{b^2}{4a}+2\prod|N_1|^2\begin{Bmatrix}k(j,l,s))\left(\theta L_{nc-1s}+\bar{\theta}L_{nc-1p}\right)\\+Bm\left(\chi T_{nc-1s}+\frac{\bar{\sigma}T_{nc-1p}}{2\mu}\right)\end{Bmatrix}. \quad (59)$$

5. MASS SPECTRA OF HEAVY QUARKONIA IN 2D SPACE-PHASE

In this section, the properties of charmonium meson and $b\bar{c}$ meson are calculated, in which the quarkonium meson have quark and antiquark masses. The following relation [4, 20] is used for determining quarkonium masses in the 2D space-phase:

$$M=m_q+m_{\bar{q}}+E_{nl}^2\to M=m_q+m_{\bar{q}}+E_{nc-ecp} \quad (60)$$

By substituting Eq. (57) into Eq. (29), the quarkonium mass in 2D space takes the following form:

$$M_{mecp}=M+2\prod\begin{Bmatrix}|N_0|^2\begin{Bmatrix}k(j,l,s))\left(\theta T_{nc-0s}+\frac{\bar{\theta}}{2\mu}T_{nc-0p}\right)\\+Bm\left(\chi T_{nc-0s}+\frac{\bar{\sigma}T_{nc-0p}}{2\mu}\right)\end{Bmatrix} & \text{for} \quad n=0\\|N_1|^2\begin{Bmatrix}k(j,l,s))\left(\theta L_{nc-1s}+\frac{\bar{\theta}}{2\mu}L_{nc-1p}\right)\\+Bm\left(\chi T_{nc-1s}+\frac{\bar{\sigma}T_{nc-1p}}{2\mu}\right)\end{Bmatrix} & \text{for} \quad n=1.\end{Bmatrix} \quad (61)$$

Where M is the ordinary masse in commutative quantum mechanics for the case of $N=2$, thus, the charmonium mass M_{c-mecp} is given by:

$$M_{c-mecp} = M_c + 2\prod \begin{cases} |N_0|^2 \begin{Bmatrix} k(j,l,s)) \left(\theta T_{nc-0s} + \frac{\bar{\theta}}{2\mu} T_{nc-0p} \right) \\ + Bm \left(\chi T_{nc-0s} + \frac{\bar{\sigma} T_{nc-0p}}{2\mu} \right) \end{Bmatrix} & \text{for} \quad n = 0 \\ |N_1|^2 \begin{Bmatrix} k(j,l,s)) \left(\theta L_{nc-1s} + \frac{\bar{\theta}}{2\mu} L_{nc-1p} \right) \\ + Bm \left(\chi T_{nc-1s} + \frac{\bar{\sigma} T_{nc-1p}}{2\mu} \right) \end{Bmatrix} & \text{for} \quad n = 1, \end{cases} \quad (62)$$

where M_c is the ordinary masses in commutative space for the case of $N = 2$. It is important to notice, the appearance of the polarization states of a quarkonium system indicates the validity of the results at high energy where the Dirac equation applied, which allowing to the validity to results of present search on the Plank's and nano scales level. If we make the limits $(\theta_k, \bar{\theta}_k) \to (0,0)$ we obtain al results of ordinary quantum mechanics.

6. CONCLUSIONS

In this article, we have investigated the solutions of the MSE for MECP. We showed the obtained degenerated spectrum for the modified studied potential depended by new discrete atomic quantum numbers: $|l-s| \le j \le |l+s|$ and m of electron, we have obtained the new masses of charmonium meson and $b\bar{c}$ meson. The new results indicate that, the MSE can be valid on the Plank's and Nano scales.

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