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Anomalous Relaxation Processes in Two-state Systems

Yu.S. Bystrik*, L.A. Denisova

Sumy State University, 2, Rimsky Korsakov St., 40007 Sumy, Ukraine

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In this paper, the biased relaxation processes in two-state systems whose structural elements evolve in accordance with a dichotomous random process are investigated. Using the continuous-time random walk approach, we derive the integral equation whose solution is the relaxation function and show that relaxation in these systems demonstrates the memory effects. Also, our attention is paid to studying the long-time behavior of the relaxation laws in the case when probability densities of the waiting times in the up and down states of the system have heavy and/or superheavy tails. From the asymptotic results it follows that relaxation of these systems to a certain equilibrium state can occur in an anomalously slow way. Finally, we perform the numerical calculations that confirm our theoretical predictions.

Keywords: Anomalous relaxation, Dichotomous process, Heavy / superheavy probability densities.

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1. INTRODUCTION

Relaxation processes describe transitions of a macroscopic system between equilibrium states and contain important information about general relaxation mechanisms. Such processes are frequently studied for the systems subjected to a generalized constant force which is randomly switched [1]. In this case, an external thermodynamic variable conjugate to the generalized force is one of the most important characteristics of the studied process. One example is the magnetic relaxation [2-4], where the external magnetic field and the magnetic moment of the system are the corresponding pair of conjugate variables. Another example is the dielectric relaxation [5-7], for which the external electric field and the electric dipole moment play the role of conjugate variables.

In addition to the classical Debye relaxation, for which change in the relaxation function $\mu(t)$ is proportional to its magnitude, i.e. the corresponding relaxation law has the form of $\mu(t) = \mu_0\exp(-t/T)$, where $\mu_0 = \mu(0)$ and T is the characteristic relaxation time, there exist many systems with anomalous, non-exponential relaxation (see [8-11] and references therein). For instance, relaxation in disordered media, glasses, dielectric relaxation in polymers is described by the stretched exponential function $\mu(t) = \mu_0\exp[-(t/T)^\gamma]$ $(0 < \gamma < 1)$, which is sometimes called the Kohlrausch-Williams-Watts law. Moreover, relaxation process for some phenomena, such as electron capture, electron-hole recombination, direct energy transfer in bodies with a complex structure at long times are well described by the exponentially-logarithmic function $\mu(t) \sim \exp[-\gamma\ln^\beta(t/T)]$ (γ, β are the positive parameters). This type of relaxation is slower than the Kohlrausch-Williams-Watts law, and at $\beta = 1$ it describes an important class of the power relaxation laws $\mu(t) \sim (t/T)^{-\gamma}$. Also one observes transitions from the Kohlrausch-Williams-Watts law for small times to the power behavior for long times.

A special attention is devoted to the study of the relaxation processes for the systems, whose properties are completely determined by the properties of their structural elements (such as monomolecular magnets, chain magnets, single-domain ferromagnetic nanoparticles). The two-state systems, for which constituent elements evolve according to a dichotomous (or telegraph) process [12-14], are an important class of such systems. In [15], using the continuous-time random walk (CTRW) theory [8-10, 16-17], it is studied the asymptotic behavior of the relaxation process in the case when waiting times of the process in each of two possible states (up and down) are distributed with the same probability density. It is shown that if this density has heavy or superheavy tails, then, respectively, anomalously slow or superslow relaxation is observed in the system. In this work we continue previous studies and consider the biased relaxation in two-state systems whose waiting times in the up and down states are distributed with different heavy/superheavy densities.

2. DESCRIPTION OF THE MODEL AND BASIC EQUATIONS

We consider a system consisting of a set of identical objects, each of which for a random period of time can be in one of two possible equilibrium states. An interesting and illustrative example of such systems is the systems of uniaxial single-domain ferromagnetic nanoparticles. In each nanoparticle, magnetization vector has two equilibrium directions; however in connection with the thermal fluctuations this vector can take arbitrary directions. As a result, dynamics of the magnetization vector becomes random and can be described by the stochastic Landau-Lifshitz (or Landau-Lifshitz-Gilbert) equation. At that, the necessary statistical properties are determined using the corresponding Fokker-Plank equation for the probability density of the magnetization direction [3, 18-20]. In particular, within this approach it is possible to approximately describe the influence on the magnetic relaxation of the dipole-dipole interaction of particles [21-23] and rotating external magnetic field [24-25]. Since this method is a general one, the study of relaxation in the systems can be quite difficult, if it is necessary to take into account all possible directions of the magnetization vector. However, probability of the magnetization vector

*yurabystrik@gmail.com

directions, which differ from the equilibrium ones, decreases with decreasing temperature. Therefore, when the general probability of such directions is sufficiently small, the magnetization vector dynamics can be approximately described by a dichotomous random process. As shown below, dichotomous approximation allows to study in detail without special computational difficulties the relaxation phenomenon for a wide class of two-state systems.

2.1 Derivation of the relaxation equation

In the framework of the dichotomous approximation, the state of each structural element of the system is associated with the dichotomous process $f(t)$ (see Fig. 1). According to the definition, this process during random periods of time $\{\tau_n\}$ (n = 1, 2,...) takes values + 1 or − 1, and $f(0) = 1$ at the initial moment of time $t_0 = 0$. Waiting (or residence) times $\{\tau_n\}$ are distributed with the probability densities $p_\pm(\tau_n)$, where signs "+ / −" correspond, respectively, to the waiting times distributions in the state + 1 / − 1 (i.e. for τ_n with odd/even index n).

Now, following [15], we describe the method for obtaining the relaxation law based on the CTRW approach. We note that the proposed approach is based not on the dynamic, but on the purely probabilistic considerations. Let us introduce the relaxation function as an average of the dichotomous process $f(t)$, namely

$$\mu(t) = \langle f(t) \rangle, \tag{2.1}$$

where angular brackets denote averaging over all admissible process paths. Taking into account that function $f(t)$ takes only + 1 or − 1 for each t, the definition (2.1) can be rewritten as

$$\mu(t) = \langle f(t) = +1 \rangle + \langle f(t) = -1 \rangle. \tag{2.2}$$

Accounting that $f(t)$ = + 1 for an even number of jumps (sign reversals) of the function $f(t)$ and $f(t)$ = − 1 for odd, the formula (2.2) takes the form

$$\mu(t) = \Pr\{ f(t) = +1 \} - \Pr\{ f(t) = -1 \}. \tag{2.3}$$

Here $\Pr\{\cdot\}$ determines the probability of an event in curly brackets. Introducing the probability $W_n(t)$ that the process $f(t)$ has performed exactly n (n = 1, 2,…) jumps during time t, we can write

$$\Pr\{f(t) = +1\} = \sum_{m=0}^{\infty} W_{2m}(t), \quad \Pr\{f(t) = -1\} = \sum_{m=0}^{\infty} W_{2m+1}(t). \tag{2.4}$$

If the number of jumps of $f(t)$ during time t is equal to zero (n = 0), then the first waiting time $\tau_1 > t$ and, consequently, the probability of no jump is written as

$$W_0(t) = \int_t^{\infty} d\tau_1 p_+(\tau_1). \tag{2.5}$$

For $n \geq 1$, the probabilities corresponding to n jumps of the function $f(t)$ in the interval (0, t] are expressed by the following formula:

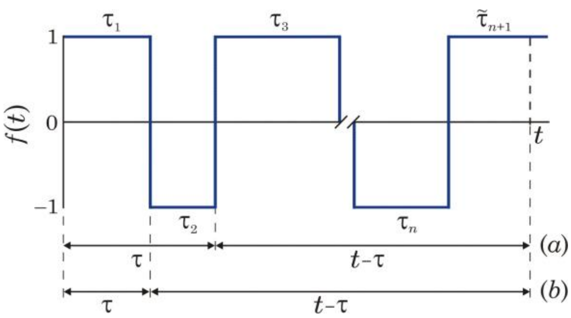

Fig. 1 – Schematic representation of the dichotomous process $f(t)$ with an even number of sign reversals in the interval (0, t]. Partition of the time interval (0, t] in the case of biased (a) and unbiased (b) relaxations are also shown here

$$W_n(t) = \int_{\Omega(t)} \left[\prod_{j=1}^{n} d\tau_j p_{(-)^{j+1}}(\tau_j) \right] \int_{\tilde{\tau}_{n+1}}^{\infty} d\tau_{n+1} p_{(-)^{n+2}}(\tau_{n+1}), \tag{2.6}$$

where domain of integration $\Omega(t)$ is determined by the condition $\sum_{j=1}^{n} \tau_j \leq t$; therefore, integral over the domain $\Omega(t)$ in (2.6) will be specified by the n-fold convolution of densities $\{p_{(-)^{j+1}}(\tau_j)\}$ (j =1, 2,..., n). Taking into account the fact that the last waiting time is equal to $\tilde{\tau}_{n+1} = t - \sum_{j=1}^{n} \tau_j$, we rewrite the expression (2.6) in expanded form

$$W_n(t) = \int_0^t d\tau_1 p_+(\tau_1) \int_0^{t-\tau_1} d\tau_2 p_-(\tau_2) \cdots \times \int_0^{t-\sum_{j=1}^{n-1}\tau_j} d\tau_n p_{(-)^{n+1}}(\tau_n) \int_{t-\sum_{j=1}^{n}\tau_j}^{\infty} d\tau_{n+1} p_{(-)^{n+2}}(\tau_{n+1}). \tag{2.7}$$

Then, applying to the equation (2.3) the Laplace transform, which for the function $g(t)$ is specified by the formula $g^s = \int_0^\infty dt\, g(t)\, \mathrm{e}^{-st}$ [Re $s > 0$], with account of (2.4) we have

$$\mu^s = \sum_{m=0}^{\infty} W_{2m}^s - \sum_{m=0}^{\infty} W_{2m+1}^s. \tag{2.8}$$

Now, using the rule of the Laplace transform for the convolution of functions, namely, $g_1^s g_2^s = \int_0^\infty dt\, \mathrm{e}^{-st} \int_0^\infty dt' \times g_1(t) g_1(t-t')$, it is not difficult to obtain the Laplace transforms for the probabilities given by the expressions (2.5) and (2.7)

$$W_{2m}^s = (p_+^s p_-^s)^m \frac{1-p_+^s}{s}, \quad W_{2m+1}^s = (p_+^s p_-^s)^m p_+^s \frac{1-p_-^s}{s}, \tag{2.9}$$

(m = 0, 1,…). Finally, substituting the formulas (2.9) into the equation (2.8) and summing the appearing infinite geometrical progressions, we find the Laplace transform for the relaxation law

$$\mu^s = \frac{1}{1 - p_+^s p_-^s} \left(\frac{1-p_+^s}{s} - p_+^s \frac{1-p_-^s}{s} \right). \tag{2.10}$$

Applying again the properties of the convolution of functions, it follows from the last formula that the relaxation function satisfies the integral equation

$$\mu(t)-\int_0^t d\tau\mu(\tau)\int_0^{t-\tau} d\tau' p_+(\tau')p_-(t-\tau-\tau') = \int_t^\infty d\tau p_+(\tau)-\int_0^t d\tau p_+(\tau)\int_{t-\tau}^\infty d\tau' p_-(\tau'). \quad (2.11)$$

It is easy to see that if the unbiased relaxation takes place, i.e. $p_\pm(\tau)=p(\tau)$, then equations (2.10) and (2.11) are simplified and take the form of [15]

$$\mu^s=\frac{1}{1+p^s}\frac{1-p^s}{s}, \quad (2.12)$$

$$\mu(t)+\int_0^t d\tau\mu(\tau)p(t-\tau)=\int_t^\infty d\tau p(\tau). \quad (2.13)$$

Equations (2.11) and (2.13) are the Volterra integral equations of the second kind with difference kernel [26], which play an important role in many renewal theory problems [27-29]. Since numerical methods of their solution are well known [30, 31], equations (2.11) and (2.13) are convenient for the numerical study of the behavior of the function $\mu(t)$ at finite times. And it is reasonable to proceed from the considerations (2.10) and (2.12) in the analytical study of the $\mu(t)$ behavior in the case of long times (see Section 3).

We consider an important class of exponential probability densities of waiting times, i.e. $p_\pm(\tau)=\bar{\tau}_\pm^{-1}e^{-t/\bar{\tau}_\pm}$, where parameter $\bar{\tau}_\pm$ is equal to the average waiting time of the system in the up/down state: $\bar{\tau}_\pm=\int_0^\infty d\tau\tau p_\pm(\tau)$. It is easy to verify that in this case the integral equation (2.11) is reduced to the differential one

$$\frac{d}{dt}\mu(t)+\left(\frac{1}{\bar{\tau}_+}+\frac{1}{\bar{\tau}_-}\right)\mu(t)+\left(\frac{1}{\bar{\tau}_+}-\frac{1}{\bar{\tau}_-}\right)=0 \quad (2.14)$$

with the initial condition $\mu(0)=1$. Its solution has a simple form

$$\mu(t)=(1-\mu_\infty)e^{-t/T}+\mu_\infty, \quad (2.15)$$

where $\mu_\infty=(\bar{\tau}_+-\bar{\tau}_-)/(\bar{\tau}_++\bar{\tau}_-)$, $T=\bar{\tau}_+\bar{\tau}_-/(\bar{\tau}_++\bar{\tau}_-)$. Based on other considerations and under certain assumptions about the relaxation process behavior, this formula is obtained in the works [21, 22, 24, 25] on study (in particular) the magnetic moment relaxation in the systems of uniaxial nanoparticles in the rotating magnetic field. In these works, the average times $\bar{\tau}_\pm$ are the parameter functions of the studied system, which are obtained using the corresponding Fokker-Plank equation. It is seen from the equations (2.10)-(2.13) that the relaxation laws can significantly differ for different probability densities $p_\pm(\tau)$ with the same average value (under the condition that it is finite). Thus, existence of the average waiting time of the system in the up and down states does not guarantee that the relaxation law will be exponential. In other words, when finding the relaxation law for a certain physical system, we should take into account additional information about its behavior. Nevertheless, the limit value of $\mu(t)$ depends only on the average values $\bar{\tau}_\pm$ and is equal to $\mu(\infty)=(\bar{\tau}_+-\bar{\tau}_-)/(\bar{\tau}_++\bar{\tau}_-)$ (Section 3).

We emphasize that an important result follows from the integral equations (2.11) and (2.13): in the general case, relaxation processes in two-state systems are characterized by the strong memory effects. In other words, the state $\mu(t)$ in each moment of time depends on all the previous moments.

2.2 Alternative derivation of the relaxation equation

Then, for better understanding of the structure and physical interpretation of the integral equations (2.11) and (2.13), we will give an alternative, more direct their derivation. We first consider the case of the biased relaxation. We remind that the right side of (2.1) is the averaging (or mathematical expectation) of the process $f(t)$ over all the possible paths. The paths are characterized by that they start at time $t_0=0$ in the point $f(0)=1$ and, having performed n ($n=1,2,...$) jumps in random times $\{\tau_n\}$, they end at time t in the point $f(t)=1$ for an even number of the process jumps or in the point $f(t)=-1$ for an odd number. If there is no jump ($n=0$), then it is obvious that $f(t)=1$. Thus, averaging of the process $f(t)$ over all the realizations can be represented as a sum of averagings over realizations for which $n=0$, $n=1$, and $n\geq 2$. So, the relaxation law $\mu(t)$ can be written as

$$\mu(t)=\left\langle f(t)\,|_{\tau_1>t}\right\rangle+\left\langle f(t)\,|_{\tau_1+\tau_2>t}\right\rangle+\left\langle f(t)\,|_{\tau_1+\tau_2\leq t}\right\rangle, \quad (2.16)$$

where the value of $\langle f(t)|_A\rangle$ denotes the mathematical expectation of the process $f(t)$ under the condition that event A has been completed.

Based on the previous results (formulas (2.2)-(2.7)), it directly follows that

$$\left\langle f(t)\,|_{\tau_1>t}\right\rangle=W_0(t)=\int_t^\infty d\tau_1 p_+(\tau_1) \quad (2.17)$$

and

$$\left\langle f(t)\,|_{\tau_1+\tau_2>t}\right\rangle=-W_1(t)=-\int_0^t d\tau_1 p_+(\tau_1)\int_{t-\tau_1}^\infty d\tau_2 p_-(\tau_2). \quad (2.18)$$

In the case when $n\geq 2$, we divide the time interval $(0, t]$ into parts $(0, \tau]$ and $(\tau, t]$, where $\tau=\tau_1+\tau_2$ (Fig. 1a). Then, the process $f(t)$ starting at time $\tau=t$ can be considered as a new process $f(t)|_{t0=\tau}$ for which the start point is shifted by τ to the right along the t axis, the initial value is equal to $f(t_0)|_{t0=\tau}=1$, and duration of a walk is $t-\tau$. Statistical properties of this process are the same as for the process $f(t-\tau)$, since they are determined only by its value at start, duration of a walk, and waiting times distributions in the up and down states (obviously the processes $f(t)|_{t0=\tau}$ and $f(t-\tau)$ are characterized by the same waiting times distributions). Hence it follows that all the statistical characteristics of the processes $f(t)|_{t0=\tau}$ and $f(t-\tau)$ are equal including the equality of their average values

$$\left\langle f(t)\,|_{t_0=\tau}\right\rangle=\left\langle f(t-\tau)\right\rangle. \quad (2.19)$$

We emphasize that, certainly, the processes $f(t)|_{t0=\tau}$ and $f(t-\tau)$ themselves are not identical (with the only difference that the start time is shifted). However, taking in the general case different values and having different waiting times $\{\tau_n\}$, these processes are equivalent in the statistical sense.

For a fixed τ, the value of $f(t)$ under the condition $n \geq 2$ (we denote this quantity $f(t)|_{\tau_1 + \tau_2 = \tau}$) is equal to the value of the function $f(t)|_{t_0 = \tau}$. At that, the average value of the process $f(t)|_{\tau_1 + \tau_2 = \tau}$ over realizations is equal to the average value of $f(t)|_{t_0 = \tau}$ multiplied by the probability that the second jump of the function $f(t)$ occurred in the range of $(\tau, \tau + d\tau)$ at $d\tau \to 0$. Hence, assuming that the densities of random quantities τ_1, τ_2 are equal to $p_+(\tau_1)$, $p_-(\tau_2)$, respectively, and taking into account the circumstance that the probability density of a random quantity $\tau = \tau_1 + \tau_2$ is equal to $\int_0^\tau d\tau' p_+(\tau') p_-(\tau - \tau')$ [29], we obtain

$$\left\langle f(t)\,|_{\tau_1+\tau_2=\tau} \right\rangle = d\tau \int_0^\tau d\tau'\, p_+(\tau') p_-(\tau-\tau') \left\langle f(t)\,|_{t_0=\tau} \right\rangle. \quad (2.20)$$

This formula with account of the equality (2.19) takes the form

$$\left\langle f(t)\,|_{\tau_1+\tau_2=\tau} \right\rangle = d\tau \int_0^\tau d\tau'\, p_+(\tau') p_-(\tau-\tau') \left\langle f(t-\tau) \right\rangle. \quad (2.21)$$

The value of $\langle f(t)|_{\tau_1 + \tau_2 \leq t} \rangle$ is the average value of the process $f(t)$ at $n \geq 2$ for all possible values of the variable τ ($\tau \in (0, t]$). Therefore, it is determined by integration of $\langle f(t)\,|_{\tau_1+\tau_2=\tau} \rangle$ with respect to all τ and thus

$$\left\langle f(t)\,|_{\tau_1+\tau_2\leq t} \right\rangle = \int_0^t d\tau \left\langle f(t-\tau) \right\rangle \int_0^\tau d\tau'\, p_+(\tau') p_-(\tau-\tau'). \quad (2.22)$$

Then, taking into consideration that $\left\langle f(t-\tau) \right\rangle = \mu(t-\tau)$ (definition (2.1)), we have

$$\left\langle f(t)\,|_{\tau_1+\tau_2\leq t} \right\rangle = \int_0^t d\tau\, \mu(t-\tau) \int_0^\tau d\tau'\, p_+(\tau') p_-(\tau-\tau'). \quad (2.23)$$

Finally, substituting the formulas (2.17), (2.18), (2.23) into the equation (2.16) and also accounting that for arbitrary functions $g_1(t)$ and $g_2(t)$ the equality of their convolutions $\int_0^t dt g_1(t-\tau) g_2(\tau) = \int_0^t dt g_1(\tau) g_2(t-\tau)$ holds, we get to the integral equation (2.11).

After implementing the direct derivation of this equation, it is not difficult to give its interpretation: the relaxation function $\mu(t)$ is equal to the difference of the probability that the process $f(t)$ did not make a single jump and the probability that the process $f(t)$ made exactly one jump plus the average value of the function $\mu(t)$ under the condition that the process $f(t)$ made more than one jump.

Now we obtain the integral equation (2.13) for the unbiased relaxation. The method of derivation will be similar to the biased case with the only difference that averaging of the process $f(t)$ over all the realizations will be written as a sum of averages over realizations, for which $n = 0$ and $n \geq 1$. Thus, we present the relaxation function $\mu(t)$ as

$$\mu(t) = \left\langle f(t)\,|_{\tau_1>t} \right\rangle + \left\langle f(t)\,|_{\tau_1\leq t} \right\rangle, \quad (2.24)$$

where

$$\left\langle f(t)\,|_{\tau_1>t} \right\rangle = W_0(t) = \int_t^\infty d\tau_1 p(\tau_1). \quad (2.25)$$

To calculate $\left\langle f(t)\,|_{\tau_1\leq t} \right\rangle$, we divide the time interval (0, t] by an intermediate point $\tau = \tau_1$ into parts (0, τ] and (τ, t] (see Fig. 1b). Starting from time $t = \tau$ the process $f(t)$ can be represented as a new process $f(t)\,|_{t_0=\tau}$ with the initial condition $f(t_0)\,|_{t_0=\tau} = -1$. Therefore, the processes $f(t - \tau)$ and $-f(t)\,|_{t_0=\tau}$ will possess the same statistical properties. Repeating the previous derivation, we write the average value of $f(t)$ under the condition that the first jump was in the range of $(\tau, \tau + d\tau)$

$$\left\langle f(t)\,|_{\tau_1=\tau} \right\rangle = d\tau p(\tau) \left\langle f(t)\,|_{t_0=\tau} \right\rangle \quad (2.26)$$

or

$$\left\langle f(t)\,|_{\tau_1=\tau} \right\rangle = -d\tau p(\tau) \left\langle f(t-\tau) \right\rangle. \quad (2.27)$$

The average value of $f(t)$ under the condition that at least one jump of the process has occurred during time t is given by the expression

$$\left\langle f(t)\,|_{\tau_1\leq t} \right\rangle = -\int_0^t d\tau \left\langle f(t-\tau) \right\rangle p(\tau). \quad (2.28)$$

Hence, taking into account the definition (2.1), we have

$$\left\langle f(t)\,|_{\tau_1\leq t} \right\rangle = -\int_0^t d\tau\, \mu(t-\tau) p(\tau). \quad (2.29)$$

Eventually, substituting the expressions (2.25) and (2.29) into (2.24) and accounting the property of the convolution of functions, we come to the integral equation (2.13).

This equation also has a simple interpretation: in each moment of time the value of the relaxation function $\mu(t)$ is equal to the difference of the probability that the process $f(t)$ did not make a single jump and the average value of $\mu(t)$ under the condition that at least one jump of the process $f(t)$ has occurred. We note that strong memory effects in this model are determined by the integral terms in the equations (2.13) and (2.11). Their presence is associated with the fact that one (for the unbiased) or two (for the biased) first jumps of the process $f(t)$ can occur at any moment of time, and therefore, calculating the value of the relaxation function $\mu(t)$, we should take into account all the possible times of these first jumps.

In the next Section, we show that for a wide class of the probability densities of the relaxation waiting times in two-state systems will be anomalously slow for long times.

3. ASYMPTOTIC BEHAVIOR OF THE RELAXATION LAWS

We consider the behavior of the relaxation law in two-state systems at large values of time for a class of distributions $p_\pm(\tau)$ with heavy and superheavy tails. The choice of this class of functions is conditioned by the fact that they play a great role in the systems with anomalous behavior [8-10, 32-35]. Then we will study the biased case [$p_+(\tau) \neq p_-(\tau)$]; the unbiased situation [$p_+(\tau) = p_-(\tau)$], as previously stated, is considered in [15]. We emphasize that in this Section variable τ is coupled only with the distribution of random times $\{\tau_n\}$ and does not concern the division of the time interval (0, t] from the previous Section.

3.1 Heavy tails $p_{\pm}(\tau)$

The probability densities $p_{\pm}(\tau)$ are called heavy if the asymptotic behavior of their tails is determined by the formula

$$p_{\pm}(\tau) \sim \frac{q_{\pm}}{\tau^{1+\alpha_{\pm}}} \quad (3.1)$$

at $\tau \to \infty$, where $q_{\pm} > 0$ and tail parameter $\alpha_{\pm} \in (0, 2]$. The restriction on the values of $\alpha_{\pm}$ is due to the fact that in this case all the moments (including fractional) of the order of $\eta \geq \alpha_{\pm}$ for the densities $p_{\pm}(\tau)$ will be infinite. So, these densities always have an infinite dispersion; the average value is infinite only if $\alpha_{\pm} \in (0, 1]$.

According to the Tauberian theorem of Karamata [9, 29], the behavior of $\mu(t)$ for large t will be specified by the behavior of μ^s for small s. More precisely, this theorem says that if, starting from a certain point, the function $k(t)$ is monotonic and

$$k^s \sim \frac{1}{s^{\gamma}} L\left(\frac{1}{s}\right) \quad (3.2)$$

at $s \to 0$, then

$$k(t) \sim \frac{t^{\gamma-1}}{\Gamma(\gamma)} L(t) \quad (3.3)$$

at $t \to \infty$. Here $\mathrm{Re}\, s > 0$, $\gamma > 0$, $\Gamma(\cdot)$ is the gamma-function, and $L(\cdot)$ is the function slowly varying at infinity, i.e. $L(\nu t) \sim L(t)$ at $t \to \infty$ for all $\nu > 0$.

Equation (2.10) shows that the behavior of μ^s at $s \to 0$ is determined by the behavior of $p_{\pm}^s$ at $s \to 0$. We will rewrite it in the form

$$\mu^s = \frac{\phi_+^s - \phi_-^s + \phi_+^s \phi_-^s}{s(\phi_+^s + \phi_-^s - \phi_+^s \phi_+^s)}, \quad (3.4)$$

where the notation $\phi_{\pm}^s = 1 - p_{\pm}^s$ is introduced. It is known that for the densities with the asymptotic behavior determined by (3.1) the following formulas (see, for example, [36]) are valid:

$$\phi_{\pm}^s \sim \begin{cases} \dfrac{q_{\pm}\Gamma(1-\alpha_{\pm})}{\alpha_{\pm}} s^{\alpha_{\pm}}, & \alpha_{\pm} \in (0,1), \\ q_{\pm} s \ln(1/s), & \alpha_{\pm} = 1, \\ \bar{\tau}_{\pm} s - \dfrac{q_{\pm}\Gamma(2-\alpha_{\pm})}{\alpha_{\pm}(\alpha_{\pm}-1)} s^{\alpha_{\pm}}, & \alpha_{\pm} \in (1,2), \\ \bar{\tau}_{\pm} s - \dfrac{q_{\pm}}{2} s^2 \ln(1/s), & \alpha_{\pm} = 2. \end{cases} \quad (3.5)$$

Here, as well as in the previous Section, $\bar{\tau}_{\pm}$ is the average waiting time of the system in the up/down state. Using the presented above results, it is possible to find the asymptotic relaxation laws at all the distributions $p_{\pm}(\tau)$ with heavy tails. We note that only tails $p_{\pm}(\tau)$ influence the asymptotic behavior of $\mu(t)$; and it does not matter how the distributions $p_{\pm}(\tau)$ behave at finite τ. Further, we perform the analysis for the case $\alpha_+ < \alpha_-$ (i.e. $p_+(\tau) \gg p_-(\tau)$ at large τ). If $\alpha_+ > \alpha_-$, then in all the obtained formulas it is just necessary to interchange indices "+" and "–", and also to replace function $\mu(t)$ by $-\mu(t)$. The situation with $\alpha_+ = \alpha_-$ (but $p_+(\tau) \neq p_-(\tau)$) is not considered, since in this case one needs to know additional terms of expansion $p_{\pm}(\tau)$ at infinity.

1) $\alpha_+ \in (0, 1]$.

If $\alpha_+ \in (0, 1]$, then it follows from (3.4) with account of (3.5) that $\mu^s \sim 1/s$ ($s \to 0$) and therefore $\mu(t) \sim 1$ ($t \to \infty$). To find the character of tendency of $\mu(t)$ to unity, we will consider the auxiliary function $h(t) = 1 - \mu(t)$, for which $h^s \sim 1/s - \mu^s$. Thus, we obtain

$$h^s = \frac{2(\phi_-^s - \phi_+^s \phi_-^s)}{s(\phi_+^s + \phi_-^s - \phi_+^s \phi_+^s)} \quad (3.6)$$

and hence we have at $s \to 0$

$$h^s \sim \frac{2\phi_-^s}{s\phi_+^s}. \quad (3.7)$$

Since $\alpha_+ \in (0, 1]$ and $\alpha_+ < \alpha_-$, then the following situations are possible in this case: (a) $\alpha_{\pm} \in (0, 1)$; (b) $\alpha_+ \in (0, 1)$ and $\alpha_- = 1$; (c) $\alpha_+ \in (0, 1)$ and $\alpha_- \in (1, 2]$; (d) $\alpha_+ = 1$ and $\alpha_- \in (1, 2]$. Using (3.5), (3.7) and the Tauberian theorem (formulas (3.2), (3.3)), we obtain, respectively, for (a), (b), (c), and (d) the asymptotics of the relaxation law

$$\mu(t) \sim 1 - \frac{2q_-\alpha_+\Gamma(1-\alpha_-)}{q_+\alpha_-\Gamma(1-\alpha_+)\Gamma(1+\alpha_+-\alpha_-)} t^{\alpha_+-\alpha_-}, \quad (3.8)$$

$$\mu(t) \sim 1 - \frac{2q_-\alpha_+ \sin(\pi\alpha_+)}{\pi q_+} \ln(t) t^{\alpha_+-1}, \quad (3.9)$$

$$\mu(t) \sim 1 - \frac{2\bar{\tau}_-\alpha_+ \sin(\pi\alpha_+)}{\pi q_+} t^{\alpha_+-1}, \quad (3.10)$$

$$\mu(t) \sim 1 - \frac{2\bar{\tau}_-}{q_+} \frac{1}{\ln(t)}. \quad (3.11)$$

2) $\alpha_+ \in (1, 2)$.

Now we will examine the asymptotic behavior of the relaxation law at $\alpha_+ \in (1, 2)$ (and therefore $\alpha_- \in (1, 2]$). We should emphasize that we do not yet consider the case $\alpha_+ = 2$, since $\alpha_+ < \alpha_-$ and this entails the fulfillment of the condition $\alpha_- > 2$ (and, as a result, dispersion $p_-(\tau)$ would be finite). Substituting the necessary formulas from (3.5) into the expression (3.4), we come to the conclusion that $\mu^s \sim [(\bar{\tau}_+ - \bar{\tau}_-)/(\bar{\tau}_+ + \bar{\tau}_-)](1/s)$ $(s \to 0)$ and, thus, $\mu(t) \sim (\bar{\tau}_+ - \bar{\tau}_-)/(\bar{\tau}_+ + \bar{\tau}_-)$ $(t \to \infty)$. Consequently, in order to determine the character of tendency of $\mu(t)$ to a limiting value, it is reasonable to introduce the auxiliary function $h(t) = (\bar{\tau}_+ - \bar{\tau}_-)/(\bar{\tau}_+ + \bar{\tau}_-) - \mu(t)$, for which the Laplace transform has the form

$$h^s = \frac{2(-\bar{\tau}_-\phi_+^s + \bar{\tau}_+\phi_-^s - \bar{\tau}_+\phi_+^s\phi_-^s)}{s(\bar{\tau}_+ + \bar{\tau}_-)(\phi_+^s + \phi_-^s - \phi_+^s\phi_+^s)}. \quad (3.12)$$

Similarly to the previous case, using the Tauberian theorem (3.2), (3.3) and the corresponding asymptotics of $\phi_\pm^s$ (3.5), we obtain

$$\mu(t) \sim \frac{\bar{\tau}_+ - \bar{\tau}_-}{\bar{\tau}_+ + \bar{\tau}_-} - \frac{2\bar{\tau}_- q_+}{(\bar{\tau}_+ + \bar{\tau}_-)^2 \alpha_+(\alpha_+ - 1)} t^{1-\alpha_+}. \quad (3.13)$$

We should note that in the situation when only one probability density, say $p_-(\tau)$ (since we agreed to study relaxation at $p_+(\tau) \gg p_-(\tau)$ for large τ), has a finite dispersion and the average value equal to $\bar{\tau}_-$, and tails $p_+(\tau)$ are heavy, then the formulas (3.10), (3.11), and (3.13) remain valid. We also emphasize that using the last asymptotic from the expression (3.5) and the auxiliary function $h(t) = \int_0^t d\tau[(\bar{\tau}_+ - \bar{\tau}_-)/(\bar{\tau}_+ + \bar{\tau}_-) - \mu(t)]$, it is not difficult to make sure that the relation (3.13) also holds for the parameter $\alpha_+ = 2$, if distributions $p_-(\tau)$ have a finite dispersion.

Within the above proposed approach, it is impossible to determine the behavior of $\mu(t)$ at long times for the dichotomous process $f(t)$, which is characterized by the densities $p_\pm(\tau)$ with finite dispersion. This is associated with the fact that in this case $\mu(t)$ will tend to the limiting value much faster than the power function; therefore, the Tauberian theorem of Karamata is not applicable. Nevertheless, in this situation using the Tauberian theorem from the equation (3.4) it is easy to establish that the relaxation function tends to the limiting value $\mu(\infty) = (\bar{\tau}_+ - \bar{\tau}_-)/(\bar{\tau}_+ + \bar{\tau}_-)$.

In Fig. 2 we illustrate the relaxation law in the case when the tail parameters $\alpha_+ \in (0, 1]$ and $\alpha_- \in (1, 2]$. In order to perform the calculations we used heavy probability densities of waiting times of the following form: $p_\pm(\tau) = \alpha_\pm/(1+\tau)^{1+\alpha_\pm}$. The solution of the integral equation (2.11) is found using the known quadrature method [30, 31], and the asymptotic behavior is obtained by the formula (3.10), where (taking into account the chosen densities $p_\pm(\tau)$) the necessary parameters are specified by the equalities $q_+ = \alpha_+$ and $\bar{\tau}_- = 1/(\alpha_- - 1)$. Finally, the numerical simulation is carried out using the proposed in [35] method for $N = 10^5$ realizations of the process $f(t)$. As seen, the analytical and numerical results agree well with each other. We have also studied numerically other relaxation laws of this Subsection; however, we consider only one example to avoid excessive cumbersomeness.

3.2 Superheavy tails $p_\pm(\tau)$

Superheavy probability densities are characterized by the fact that their asymptotic behavior at $\tau \to \infty$ is described by the expression

$$p_\pm(\tau) \sim \frac{\sigma_\pm(\tau)}{\tau}. \quad (3.14)$$

In the last relation, $\sigma_\pm(\cdot)$ are the positive functions slowly varying at infinity. To provide normalization of the densities $p_\pm(\tau)$, the condition $\sigma_\pm(\tau) = o(1/\ln\tau)$ at $\tau \to \infty$ is imposed on $\sigma_\pm(\cdot)$. It is convenient to introduce the following functions for further study:

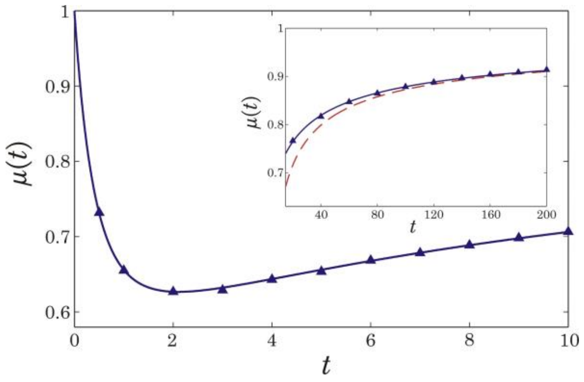

Fig. 2 – Relaxation law in the case when $\alpha_+ = 0.5$ and $\alpha_- = 1.5$. The solid blue line corresponds to the exact relaxation law $\mu(t)$ obtained using the solution of the integral equation (2.11); blue triangles correspond to the numerical simulation results; the dashed red line corresponds to the asymptotic formula (3.10)

$$V_\pm(t) = \int_t^\infty d\tau p_\pm(\tau), \quad (3.14)$$

equal to the probability that waiting times of the process $f(t)$ in the up/down state exceed t. Using the L'Hospital rule, it is easy to show that $\lim_{t\to\infty} V_\pm(\nu t)/V_\pm(t) = 1$, i.e. functions $V_\pm(t)$ also vary slowly. Then, using the properties of slowly varying functions [37, 38], we obtain

$$\phi_\pm^s = sV_\pm^s = \int_0^\infty d\tau e^{-\tau} V_\pm(\tau/s) \sim V_\pm(1/s). \quad (3.15)$$

As well as for heavy tails, we perform the calculations at $p_+(\tau) \gg p_-(\tau)$ (otherwise, in the formulas obtained below it is just necessary to interchange indices "+" and "−", and the function $\mu(t)$ should be replaced by $-\mu(t)$). Since $V_+(1/s) \gg V_-(1/s)$ at $s \to 0$, it follows from (3.15) and (3.4) that $\mu(t) \sim 1$ at long times. Thus, we again come to the need to use the auxiliary function $h(t) = 1 - \mu(t)$ and the asymptotic relation (3.7). Substituting the formula (3.15) into it and applying the Tauberian theorem (3.2), (3.3), we obtain $h^s \sim 2V_-(1/s)[sV_+(1/s)]^{-1}$, from which we find the relaxation law

$$\mu(t) \sim 1 - \frac{2V_-(t)}{V_+(t)}. \quad (3.16)$$

We now study the situation when the tail of only one probability density, say $p_+(\tau)$, is superheavy. In this case, it is necessary to separately examine three different behavior patterns of $p_-(\tau)$: (a) heavy tail with $\alpha_- \in (0,1)$; (b) heavy tail with $\alpha_- = 1$; (c) $p_-(\tau)$ has a finite average value $\bar{\tau}_-$ (it is obvious that this includes both the heavy distributions $p_-(\tau)$ with $\alpha_- \in (1,2]$ and the distributions with finite dispersion). For the case (a), it is enough to substitute the necessary asymptotics (3.5), (3.15) into the formula (3.7) to obtain based on the Tauberian theorem

$$\mu(t) \sim 1 - \frac{2q_- t^{-\alpha_-}}{\alpha_- V_+(t)}. \quad (3.17)$$

In the situations (b) and (c), we determine the auxiliary function as $h(t) = \int_0^t d\tau[1 - \mu(t)]$, for which at small s the following relation holds:

$$h^s \sim \frac{2\phi_-^s}{s^2\phi_+^s}. \tag{3.18}$$

Hence, for the case (b) we obtain

$$\mu(t) \sim 1 - \frac{2q_-}{V_+(t)t}\left[1 + \frac{t\ln(t)p_+(t)}{V_+(t)}\right], \tag{3.19}$$

and for the case (c)

$$\mu(t) \sim 1 - \frac{2\overline{\tau}_- p_+(t)}{V_+^2(t)}. \tag{3.20}$$

We should note that two time-dependent terms of the asymptotic expansion of $\mu(t)$ appear in the formula (3.19). This is associated with the fact that for different probability densities $p_+(\tau)$ each of these terms can make a major contribution to the expression $1 - \mu(t)$ or both have the same order of smallness. For example, if chose the distributions $p_+(\tau)$ such that the probability of no jump is equal to $V_+(t) = 1/\ln\ln t$, $1/\ln t$, and $\exp(-\ln t/\ln\ln t)$, then for the equation (3.19) a major contribution in square brackets will be determined by the first, both (since the second term at infinity tends to unity) and the second term, respectively.

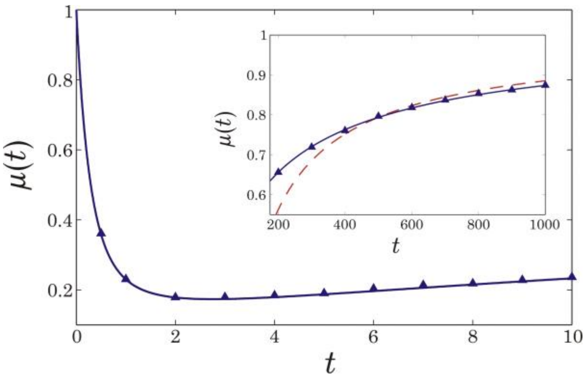

Fig. 3 – Relaxation law in the case when density $p_+(\tau)$ has a superheavy tail, and $p_-(\tau)$ – a heavy tail with $\alpha_- = 1.5$. The solid blue line corresponds to the exact relaxation law $\mu(t)$ obtained using the solution of the integral equation (2.11); blue triangles show the numerical simulation results; and the dashed red line corresponds to the asymptotic formula (3.20)

In Fig. 3 we present the example of the behavior of the relaxation law in the case, when the probability density $p_+(\tau)$ is characterized by a superheavy tail, and the density $p_-(\tau)$ – by a heavy tail with the parameter $\alpha_- \in (1, 2]$. For the numerical calculations we took $p_+(\tau)$ of the form $p_+(\tau) = \nu \ln^\nu g(g + \tau)^{-1} \ln^{-1-\nu}(g + \tau)$ with the parameters $\nu = g = 2$ and density $p_-(\tau)$ of the same form that in the Subsection 3.1 with $\alpha_- = 1.5$. The solution of the integral equation (2.11) and the numerical simulation method are also similar to those from Subsection 3.1.

It follows from the obtained asymptotic relaxation laws that relaxation in all the considered cases exhibits anomalous properties. At that, the spectrum of behavior of the relaxation law is very wide and depends significantly on the asymptotics of the waiting times distribution tails in the up and down states of the dichotomous process. Thus, for the considered type of random systems, relaxation to the equilibrium state can behave as a slow power function, superslow (slowly varying) function and also as a product of the power and slowly varying functions. All these three classes of functions belong to the so-called correctly varying functions [37, 38].

4. CONCLUSIONS

Using the CTRW theory, we have studied the relaxation process for a wide class of two-state systems whose structural elements vary according to the dichotomous process. In this work we have obtained the integral equation for the relaxation law in the case of arbitrary probability densities of waiting times of the system in the up and down states. It follows from the integral equation that in the general case for two-state systems the relaxation process is non-local in time and possesses strong memory effects. Assuming that the probability densities of waiting times in the up and down states have heavy and/or superheavy tails and using the Laplace transform and the Tauberian theorem of Karamata, we have found the asymptotic behavior of the studied relaxation laws for long times and shown that the corresponding relaxation laws are anomalous. The obtained relaxation laws are universal for two-state systems of any nature and differ qualitatively from the earlier known. The analytical results are confirmed by the numerical calculations.

REFERENCES

1. S. Dattagupta, *Relaxation Phenomena in Condensed Matter Physics* (Academic Press: Orlando: 1987).
2. B. Cowan, *Nuclear Magnetic Resonance and Relaxation* (Cambridge University Press: Cambridge: 2005).
3. H. Suhl, *Relaxation Processes in Micromagnetics* (Oxford University Press: Oxford: 2007).
4. G. Bertotti, I. Mayergoyz, C. Serpico, *Nonlinear Magnetization Dynamics in Nanosystems* (Elsevier: Oxford: 2009).
5. K.C. Kao, *Dielectric Phenomena in Solids* (Elsevier Academic Press: San Diego: 2004).
6. R.H. Boyd, G.D. Smith, *Polymer Dynamics and Relaxation* (Cambridge University Press: Orlando: 2007).
7. K.L. Ngai, *Relaxation and Diffusion in Complex Systems* (Springer: New York: 2011).
8. R. Klages, G. Radons, I.M. Sokolov, eds., *Anomalous Transport: Foundations and Applications* (Wiley-VCH: Berlin: 2008).
9. B.D. Hughes, *Random Walks and Random Environments,* Vol. **1** (Clarendon Press: Oxford: 1995).
10. R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).
11. A. Blumen, J. Klafter, G. Zumofen, in: I. Zschokke (Ed.), *Optical Spectroscopy of Glasses* (Reidel: Dordrecht: 1986).
12. S. Goldstein, *Quart. J. Mech. Appl. Math.* **4**, 129 (1951).
13. M. Kac, *Rocky Mount. J. Math.* **4**, 497 (1974).
14. A.D. Kolesnik and N. Ratanov, *Telegraph Processes and Option Pricing* (Springer: Heidelberg: 2013).
15. S.I. Denisov, Yu.S. Bystrik, *Acta Phys. Pol. B* **46**, 931 (2015).
16. E.W. Montroll, G.H. Weiss, *J. Math. Phys.* **6**, 167 (1965).
17. D. ben-Avraham, S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press: Cambridge: 2000).
18. J.L. Dormann, D. Fiorani, E. Tronc, *Adv. Chem. Phys.* **98**, 283 (1997).
19. W.T. Coffey, Yu.P. Kalmykov, J.T. Waldron, *The Langevin Equation, 2nd ed.* (World Scientific: Singapore: 2004).
20. W.T. Coffey, Yu.P. Kalmykov, *J. Appl. Phys.* **112**, 121301 (2012).

21. S.I. Denisov, K.N. Trohidou, *Phys. Rev. B* **64**, 184433 (2001).
22. S.I. Denisov, T.V. Lyutyy, K.N. Trohidou, *Phys. Rev. B* **67**, 014411 (2003).
23. P.M. Déjardin, *J. Appl. Phys.* **110**, 113921 (2011).
24. S.I. Denisov, T.V. Lyutyy, P. Hänggi, K.N. Trohidou, *Phys. Rev. B* **74**, 104406 (2006).
25. S.I. Denisov, K. Sakmann, P. Talkner, and P. Hänggi, *Phys. Rev. B* **75**, 184432 (2007).
26. A.D. Polyanin, A.V. Manzhirov, *Handbook of Integral Equations*, *2nd ed.* (Chapman & Hall/CRC Press: London: 2008).
27. D.R. Cox, *Renewal Theory* (Methuen and Co.: London: 1967).
28. W.L. Smith, *J. Roy. Statist. Soc. Ser. B* **20**, 243 (1958).
29. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2 (Wiley: New York: 1971).
30. K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind* (Cambridge University Press: Cambridge: 1997).
31. L.M. Delves, J.L. Mohamed, *Computational Methods for Integral Equations* (Cambridge University Press: Cambridge: 1985).
32. S.I. Denisov, H. Kantz, *Europhys. Lett.* **92**, 30001 (2010).
33. S.I. Denisov, H. Kantz, *Phys. Rev. E* **83**, 041132 (2011).
34. S.I. Denisov, S.B. Yuste, Yu.S. Bystrik, H. Kantz, K. Lindenberg, *Phys. Rev. E* **84**, 061143 (2011).
35. S.I. Denisov, Yu.S. Bystrik, H. Kantz, *Phys. Rev. E* **87**, 022117 (2013).
36. S.I. Denisov, E.S. Denisova, H. Kantz, *Eur. Phys. J. B* **76**, 1 (2010).
37. N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation* (Cambridge University Press: Cambridge: 1987).
38. E. Seneta, *Functions of regular variation. Lecture Notes in Mathematics* **508** (Springer-Verlag: New York: 1976).