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INTERACTION OF E-POLARIZED WAVE WITH PREFRACTAL WEAKLY FILLED DIFFRACTION GRATING (AN ASYMPTOTICAL MODEL)

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An asymptotical model of E-polarized electromagnetic wave interaction with weakly filled prefractal diffraction grating (PFDG) is considered in detail on the base of rigorous electromagnetic theory. A stage of construction for Cantor set with variable Hausdorff dimension is used for PFDG order. An integral equation technique with usage of asymptotical approach and Carleman inversion formula is applied. Asymptotical formulas for determination of the main electromagnetic characteristics are obtained. Numerical experiments are done to find the fractal properties of the prefractal grating.

***Keywords**: ELECTROMAGNETIC WAVES, FRACTAL MODELING, DIFFRACTION GRATINGS, INTEGRAL EQUATIONS.*

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1. INTRODUCTION

In due time, the theory of periodic diffraction gratings (DG) of the strip type and different modifications has been thoroughly developed by the school of Kharkov radiophysicists [1-3]. Investigation of the so-called "fractal" DG was started with the appearance of the theory of fractals, but the term "fractal" does not have a clear definition up to now and is interpreted differently. It is possible to single out from the variety of "fractal" objects a rather wide class of the so-called self-similar fractals, whose clear definition is the following: this is a set with the Hausdorff dimension (HD) which is larger than the topological one [4].

In the present work we consider PFDG in the form of the system of strips placed in accordance with the segments, which form a certain stage of the construction of the perfect Cantor set (PCS) with variable fractal dimension [5]. We have to note that the introduced term "prefractal" DG foresees not an infinite sequence of DG elements according to the rule of the chosen fractal, but, first of all, their limited amount. Therefore, this term can be also interpreted as the truncated, reduced, shortcut fractal, almost fractal or quasi-fractal. In this case formulation of the electrodynamic problem is classically rigorous within certain traditional assumptions [6]. The aim of the present paper is the solution of the boundary-value problem of diffraction of a plane electromagnetic wave on a finite prefractal sequence of grating elements and the detailed analysis of scattered fields for the asymptotic case of the model of weakly filled DG.

2. STATEMENT OF THE PROBLEM

A plane electromagnetic wave arrives to a system with certain finite amount of infinitely thin and perfectly conducting cylinder strips with parallel edges. Arrangement of the strips in the system should be strictly ordered in accordance with different mathematical laws, which correspond to the processes of the PCS construction with variable HD. Therefore, statement of the problem should be itemized with taking into account pointed new mathematical order of the strip arrangement.

Since generatrices of the strips are parallel, the Maxwell equations are devided into two independent systems of equations in separated components of the electric and magnetic fields. Their solution leads to the two-dimensional Helmholtz equation and two types of the boundary conditions on the elements of scatterer directrices. From the mathematical point of view, we have the external Dirichlet (E-polarization) and Neumann (H-polarization) problems for the mentioned equation. These problems have a unique solution when the radiation condition at infinity (by Sommerfeld) and the conditions in boundary points of the Meixner arc [7] hold.

To study the above formulated external boundary problems of mathematical physics, we use a classical method of integral equations (IE), which not only decreases the problem dimension, but also reduces the external problem of mathematical physics to the IE solution on the corresponding finite amount of smooth arcs that essentially simplifies the problem. Moreover, IE are more convenient mathematical models in comparison with the boundary or boundary-value problems for partial differential equations relative to their numerical solution by computer and asymptotical analytical solution.

In the present paper we only consider the external Dirichlet problem for two-dimensional Helmholtz equation that corresponds to the diffraction problem of a plane electromagnetic wave, whose electric field vector is parallel to generatrices of the strips (E-polarization).

Using the fundamental solution of the Helmholtz equation for two-dimensional free space and the IE method, we obtain the system of equations

$$\sum_{m=1}^{2^n} \int_{-1}^{1} j_m(t) H_0^{(1)}\left(\left|x_\ell^n(\tau) - x_m^n(t)\right|\right) dt = \frac{2i}{\pi} \exp[i \cdot q_1 \cdot x_\ell^n(\tau)], \; \ell = 1, \ldots, 2^n . \qquad (1)$$

Here $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind; functions $x_m^n(t)$ are initial variables of the PCS construction process with variable HD; q_1 is the first component of the directing vector of a plane wave. To impart to this system, which is the basic mathematical model, the scattering process, we will further consider the PCS construction process with variable HD.

3. THE PCS CONSTRUCTION PROCESS WITH VARIABLE HD

Simplicity of the initial object and iteration principle of the formation is the characteristic feature of the PCS. We start the PCS construction from the segment of the length of $2a$. From the middle of this segment we remove the interval of the length of $2b$, of course, $a > b$. We call obtained two segments the former of the given set or its first construction stage. In Fig. 1 we show the initial segment (black line on the abscissa axis) and three stages of the construction: two segments of the first stage are depicted in red.

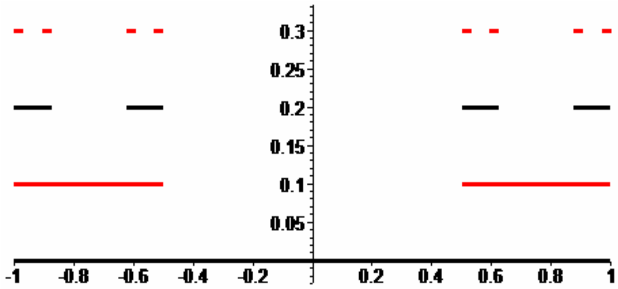

***Fig. 1** – Stages of the PCS construction with HD* 0,5

To make the object to be self-similar, we reduce former to the size of its separate segment and substitute them. In this case we obtain four segments reduced from the previous ones by $\kappa = 2a/(a-b) > 2$ times, which are shown in Fig. 1 in black on the level $y = 0{,}2$. In particular, when $b = a/3$ we have $\kappa = 3$, i.e. the self-similarity coefficient of the classical PCS.

The next step – transition to the third stage of the fractal construction – is one more reduction of the former by κ times and segment substitution of the second stage, it leads to eight segments shown in red in Fig. 1 on the level $y = 0{,}3$. If continue this process infinitely, the perfect set will be formed, which slightly differs from the classical one and is its particular case. Therefore, we can talk about whole class of perfect sets, which depend on the self-similarity coefficient κ.

Calculation of the HD in common with the classical PCS leads to the expression $\ln 2/\ln\kappa$. Really, during construction on the n-th step we have 2^n intervals of the length of $(a-b)/\kappa^{n-1}$. Then, coverage with 2^n elements will be minimal δ-coverage with $\delta = (a-b)/\kappa^{n-1}$, i.e.

$$H^s_{\frac{a-b}{\kappa^{n-1}}} = \sum_{i=1}^{2^n} \left|U_i\right|^s = 2^n \cdot \left|\frac{a-b}{\kappa^{n-1}}\right|^s = \frac{2^n}{\kappa^{(n-1)s}}(a-b)^s \cdot$$

Hence, boundary transition $\delta = (a-b)/\kappa^{n-1} \to 0$ only in the case $2/\kappa^s = 1$ gives non-zero and non-infinite value of the Hausdorff s-measure [8]. To determine the critical value s, which is the HD, we find the logarithm of the identity $2 \equiv \kappa^s$. As a result, we obtain the mentioned expression $s = \ln 2/\ln\kappa$. Since the self-similarity coefficient κ is more than 2, the HD of the perfect set is changed in the interval $(0,\ 1)$.

Due to self-similarity the construction process can be sufficiently simply formalized by linear functions that is important for definiteness of equations (1). In particular, the first stage of the construction with taking into account the normalization on the wavelength (two segments of the relative size 2α are placed on the relative distance $2(\alpha-\beta)$ from each other) can be specified by functions $x_m^1(t) = (-1)^m \beta + \alpha t$, $m = 1{,}2$. Segments of the second stage of the

PCS construction are formalized by functions $x_m^2(t) = (-1)^m(\beta + \beta_2) + \alpha_2 t$ for $m = 1,4$ and $x_m^2(t) = (-1)^m(-\beta + \beta_2) + \alpha_2 t$ for $m = 2,3$, where $\beta_2 = \beta/\kappa$, $\alpha_2 = \alpha/\kappa$, $\kappa = 1 + \beta/\alpha > 2$. For an arbitrary natural number n, which defines the stage of the construction, we have the ordered sequence of functions $x_m^n(t)$, where subscript $m = 1, \ldots, 2^n$.

System of singular IE (1) can be considered as the basic mathematical model of the scattering process of the plane E-polarized electromagnetic wave with plane prefractal grating.

4. ASYMPTOTICAL MODEL OF THE E-POLARIZED WAVE SCATTERING BY PFDG

While constructing the PCS, parameter α_n is decreased not less than doubly during transition from one stage of the construction to another. Therefore, starting from a certain stage of the construction it can be considered as small as it is necessary, and, thus, one can expect the efficiency of the asymptotical model of narrow strips or weakly filled DG.

Since $x_\ell^n(\tau) - x_m^n(t) = \rho_{\ell m} + \alpha_n(\tau - t)$, where modulus of $\rho_{\ell m} = x_\ell^n(0) - x_m^n(0)$ is the distance between strip centers with numbers ℓ and m, have neglected terms of the first order of smallness with respect to α_n, we obtain the system of singular IE with pronounced logarithmic singularity

$$\int_{-1}^{1} j_\ell(t)\ln|\tau - t|\,dt + \sum_{m=1}^{2^n} j_m R_{\ell m} = \exp[iq_1 x_\ell^n(0)], \quad \ell = 1, \ldots, 2^n. \tag{2}$$

Here $R_{\ell m} = \pi H_0^{(1)}\left(|\rho_{\ell m}|\right)/2i\ (\ell \neq m)$ are the off-diagonal and $R_{PP} = \ln(\gamma\alpha_n/2i)$ are the diagonal coefficients. Using the known Carleman inversion formula we obtain the following vector equation:

$$\vec{j}(t) = \frac{\Re_n \cdot \vec{j} - \vec{q}_n}{\pi\sqrt{1 - t^2}\,\ln 2}. \tag{3}$$

Here vector-function $\vec{j}(t)$ consists of the components of unknown functions $j_m(t)$; $\Re_n$ is the known matrix of the corresponding size, whose elements R_{Pm} are represented above; $\vec{j} = \int_{-1}^{1} \vec{j}(t)dt$; $\vec{q}_n = \vec{q}_n(\varphi_0)$ is the known column vector composed of the right sides of the system (2) $\exp[i\cos\phi_0 x_\ell^n(0)]$; φ_0 is the incidence angle of a plane wave on the grating.

To find the unknown vector $\vec{j}$, we integrate both sides of the vector equation (3), then a system of linear algebraic equations of the following matrix form will be obtained:

$$(\Re_n - \ln 2 \cdot E_n)\vec{j} = \vec{q}_n, \tag{4}$$

where E_n is the unit matrix; in this case the desired vector-function has the following form:

$$\vec{j}(x) = \vec{j} \,/\, \pi\sqrt{1 - x^2}.$$

Thus, from the mathematical point of view, the scattering problem of a plane electromagnetic wave by a system of strips, which forms PFDG, within made assumptions can be considered solved. Further we pass to the numerical experiments in order to determine one of the main characteristics – the directional pattern, which will help to reveal the fractal features of PFDG.

5. NUMERICAL ANALYSIS OF THE DIRECTIONAL PATTERNS

After mathematical solution of the problem, scattered electromagnetic field around scatterer can be represented by the function defined as a sum of integral transformations of the solution $j_P(t)$ [6]

$$\upsilon(x,y) = -\frac{i}{4}\sum_{\ell=1}^{2^n}\int_{-1}^{1} j_\ell(t) H_0^{(1)}\left(\sqrt{(x-x_\ell(t))^2+y^2}\right)dt .$$

Hence, using the known procedure we obtain scattered electromagnetic field in the far region, which is of a great physical interest [6]. Usually, during geometrical presentation of this characteristic, it is necessary to plot graphs of the coefficient [6, 7] $A(\varphi) = -0,5\sqrt{i}\sum_{\ell=1}^{2^n}\hat{j}_\ell(\cos\varphi)$, where hat above the letter denotes the Fourier transformation, which characterizes the field distribution in the far region versus the polar angle. In the case of the asymptotical model $A(\varphi) = -\sqrt{i}\,[\vec{q}_n^{\,T}(-\cos\varphi), \vec{j}\,]/2\sqrt{2\pi}$. The numerical experiment is carried out in order to reveal the fractal properties of DG. To perform this, we compare the dependences of $|A(\varphi)|$ on the polar angle for DG, which correspond to the adjacent stages of the PCS construction. In Fig. 2 we present the directional patterns of DG for the value $\varphi_0 = \pi/2$, which correspond to the first (solid line, $n = 1$) and the second stages of the PCS construction with HD = 0,2 (for all $\alpha = \beta/31$). Here we should note that the lateral dimension of DG is closed to the wavelength.

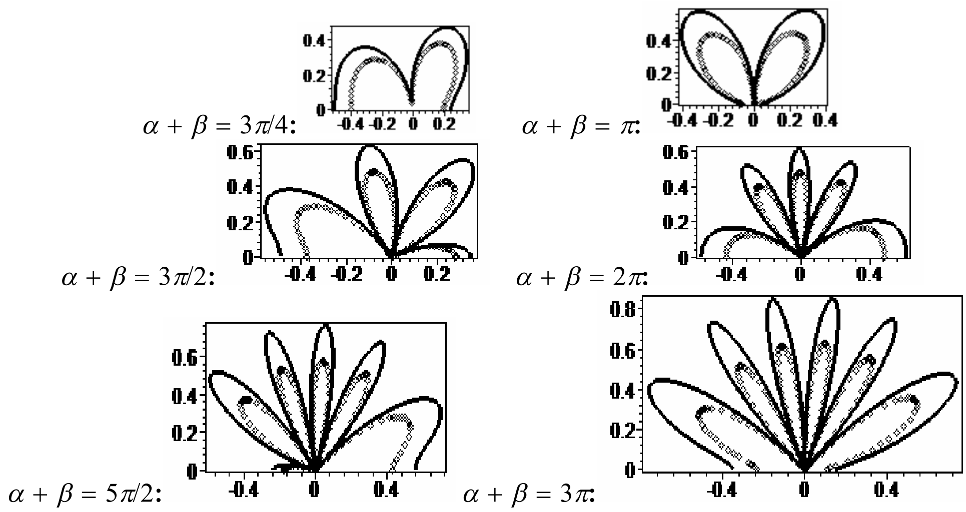

***Fig. 2** – Directional patterns of prefractal grating, which corresponds to the first and the second stages of the PCS construction ($\alpha = \beta/31$)*

Within assumptions of the asymptotical model, shapes of the directional patterns for different n are the same, though there are distinctions in sizes, because we have different filling of PFDG. If pass to the third stage of the construction, the size will be lesser but the shape will not be changed. This implies that the shape of the directional pattern is a general property for all PFDG starting from the first stage of the construction, but only for the pointed correlations between the wavelength and the initial geometric parameters. With the decrease in the wavelength, individual elements of the first stage of the construction are revealed more and more, while individual elements of the second stage of the construction are not tangible. Further decrease in the wavelength leads to the following: parameter α is not small any more and the case $n = 1$ is outside of the given asymptotical model, but parameter $\alpha_2 = \alpha/\kappa$ remains small and the case $n = 2$ still gives reliable results. Therefore, we pass to its comparison with the case $n = 3$, i.e. the next (third) stage of the PCS construction. Here individual elements of the second stage are revealed and individual elements of the third stage are not tangible. In Fig. 3 we present the directional patterns of PFDG, which correspond to the second (dark line) and the third (red line) stages of the PCS construction with the same HD 0,2 and correlation $\alpha = \beta/31$.

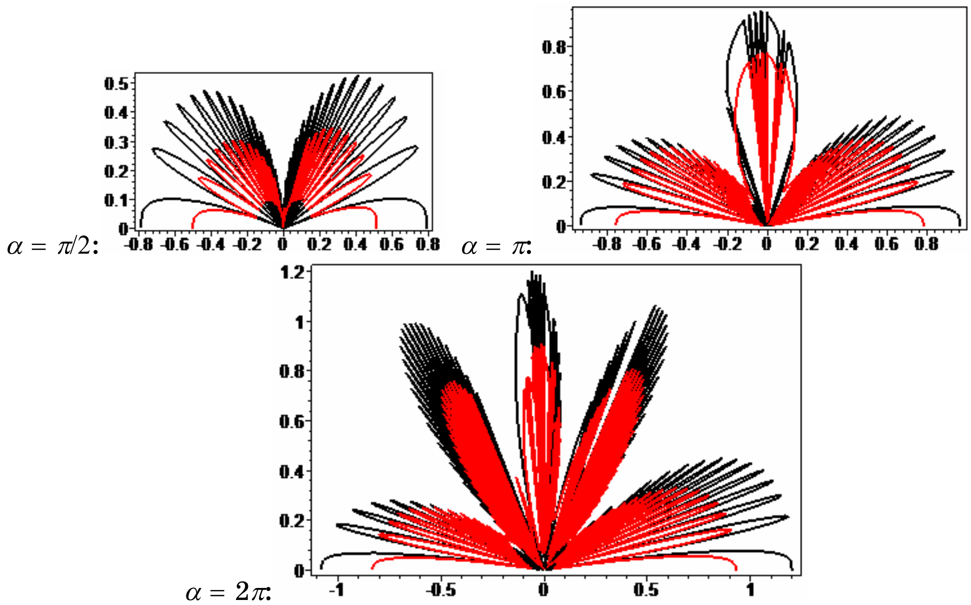

***Fig. 3** – Directional patterns of prefractal grating, which corresponds to the second and the third stages of the PCS construction ($\varphi_0 = \pi/2$)*

We note that here the lateral dimension of DG substantially exceeds the wavelength, but the grating, which corresponds to the second stage, has two rather remote sublattices (in Fig. 1 they are depicted in black on the level $y = 0{,}2$), whose lateral dimension defined by the parameter α is closed to the wavelength.

Performed numerical experiments with the calculation of one of the main DG characteristics – its directional pattern – show the possibility to reveal the fractal properties of prefractal object using this characteristic.

6. CONCLUSIONS

In the present work an asymptotical model of the interaction process of the E-polarized wave and weakly filled prefractal grating is studied in detail. For mathematical order of DG directrices it is proposed to take the certain stage of the PCS construction with variable HD, which is determined by the expression $\ln 2/\ln\kappa$. Investigation is performed based on the rigorous electromagnetic theory by the IE method using asymptotics and Carleman inversion formulas. Asymptotical expressions of the initial variables of the mathematical model, which allow to determine one of the main characteristics – the directional pattern of prefractal grating, which reveals the fractal properties of DG, are obtained.

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