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THE DYNAMICS OF NON-EQUILIBRIUM TRANSITIONS INDUCED BY THE CROSS-CORRELATED NOISES: NUMERICAL RESULTS

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The dynamic system described by the Langevin equation with two cross-correlated Gaussian white noises is considered. The non-equilibrium probability distribution function of the system is calculated by the numerical methods. The time of change of the initially unimodal distribution to the bimodal one is determined for different values of the control parameter. A critical slowing down in the transition dynamics is demonstrated.

Keywords: *LANGEVIN EQUATION, CROSS-CORRELATED WHITE NOISES, MONTE-CARLO METHOD, PROBABILITY DENSITY, NOISE-INDUCED TRANSITIONS.*

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1. INTRODUCTION

Nonlinearity and noise can result physical and other systems in the ordering. Stochastic resonance [1], directed transport [2], noise-induced non-equilibrium transitions [3], etc. are the examples of phenomena. The latter can be observed in strongly non-equilibrium open systems, which interact with a fluctuating environment. Strong non-equilibrium leads to the nonlinearity of the corresponding dynamic equations of the system, where the effect of a fluctuating environment is taken into account by the use of the noise with given statistical characteristics.

The so-called unimodal-bimodal noise-induced transition belongs to non-equilibrium transitions. It takes place if stationary probability density of the system is smoothly changed from the function with one maximum to the function with two maxima as the noise intensity smoothly increases above the critical value. In this case maxima of the probability density correspond to the system phases, maximum points – to the ordering parameter, and noise intensity – to the control parameter. That is, the mentioned transition resembles classical phase transitions and has some their peculiarities, namely, the critical indexes, critical slowing down [3].

Not only the external fluctuations, but also the internal ones induced by the thermal motion of the structural units of the system are essential for micro- and nanosystems. Their combined influence on the system can be taken into account by the use of two cross-correlated noises with known statistical characteristics. Within the approximation of two Gaussian white noises the authors of [4] investigate a relatively simple dynamic system. They find the exact expression for the equilibrium probability density of the system, show that due to the cross-correlation these noises can induce unimodal-bimodal transition, and each noise separately or non-correlated noises can not induce such transition. The critical index for the dependence of the ordering parameters on the control ones, which is equal to its classical value, is also calculated. In the present work for the mentioned dynamic system the non-equilibrium probability density is found by the numerical methods, and using it the dynamics of non-equilibrium transition induced by the cross-correlated noises is studied near the critical point.

2. THE MODEL

We consider the dynamic system described by one state parameter, which depends on the time only. The dimensionless Langevin equation in the Stratonovich calculus takes the form [4]

$$\dot{x}(t) = f(x(t)) + \sigma_1 g(x(t))\xi_1(t) + \sigma_2\xi_2(t)\,, \tag{1}$$

where $x(t)$ is the system state parameter; $x(0) = 0$; dot is the time derivative; $f(x)$ is the deterministic force; $g(x)$ is the amplitude of multiplicative noise fluctuations; $\xi_1(t)$ and $\xi_2(t)$ are the Gaussian white noises with zero mean values, intensities σ_1^2 and σ_2^2, and correlation functions

$$\langle\xi_i(t)\xi_i(t')\rangle = \delta(t-t') \;\; (i = 1,\, 2), \quad \langle\xi_1(t)\xi_2(t')\rangle = r\delta(t-t')\,;$$

angle brackets denote averaging over ensemble of noise realizations; r is the cross-correlation coefficient of the noises, $|r| \le 1$; $\delta(t)$ is the Dirac delta-function.

Eq. (1) can describe, for example, non-equilibrium processes in ballast resistor [5], directed transport of Brownian particles [6], stochastic resonance in nonlinear rotator (under the condition that function f is also explicitly time-dependent) [17], and other physical processes, where noise cross-correlation plays an important role [4] (and references in this article). Further we assume that $f(x) = -ax$, i.e., deterministic force is linear and restoring one with the parameter a ($a > 0$); $g(x) = -x^2/(1+x^2)$, amplitude of fluctuations in the vicinity of zero (the stable point in deterministic dynamics) is quadratic, and it is constant for large x. Just in this case, the system described by Eq. (1) demonstrates non-equilibrium transition induced by the cross-correlated noises as it was shown in [4]. Expression for the equilibrium probability density can be written as

$$p(x) = C\left[\left(\frac{x^2}{1+x^2}\right)^2 + 2r\nu\frac{x^2}{1+x^2} + \nu^2\right]^{-\frac{1}{2}} \exp\left[-\frac{a}{\sigma_1^2}\int_0^{x^2}\frac{dz}{\left(\frac{z}{1+z}\right)^2 + 2r\nu\frac{z}{1+z} + \nu^2}\right], \tag{2}$$

where C is the normalization constant; $\nu = -\sigma_2/\sigma_1$. Critical parameters, which determine the condition of non-equilibrium transitions, are found from the equation $r\sigma_1\sigma_2 = a$.

3. ALGORITHM OF THE NUMERICAL EXPERIMENT

Algorithm of the numerical experiment is based on the method of statistical testing (the Monte-Carlo method) [8].

We use the Fokker-Planck equation [4] and write a Langevin equation in the Ito calculus, which is statistically equivalent to the Langevin equation (1) in the Stratonovich calculus

$$\dot{x}(t) = f(x(t)) + \frac{\sigma_1^2}{2} g(x(t))g'(x(t)) + \frac{r}{2}\sigma_1\sigma_2 g'(x(t)) + \sigma_1 g(x(t))\xi_1(t) + \sigma_2\xi_2(t) ,$$

where the stroke denotes the derivative with respect to the coordinate. We apply the Euler method to the last equation, and the difference scheme for Eq. (1) takes the form

$$x_{i+1} = x_i + \left[f(x_i) + \frac{\sigma_1^2}{2} g(x_i)g'(x_i) + \frac{r}{2}\sigma_1\sigma_2 g'(x_i) \right]\Delta t + \sigma_1 g(x_i)\Delta W_{1i} + \sigma_2 \Delta W_{2i} , \quad (3)$$

where Δt is the time sampling step, $t_{i+1} = t_i + \Delta t$; ΔW_{1i} and ΔW_{2i} are the increments of the Wiener processes, whose values can be obtained from the formulas $\Delta W_{1i} = \xi_{1i}\sqrt{\Delta t}$ and $\Delta W_{2i} = \xi_{2i}\sqrt{\Delta t}$. Here ξ_{1i} and ξ_{2i} are the cross-correlated random quantities with the correlation coefficient r distributed by the normal law with zero mean value and unit dispersion. To generate the correlated values of this pair of quantities we use the formula [10]

$$\xi_{2i} = r\xi_{1i} + \sqrt{1 - r^2}\,\xi_i ,$$

where ξ_{1i} and ξ_i are the independent Gaussian quantities generated by the library functions.

Difference scheme (3) is used to find N random realizations of the system state parameter $x(t)$ in the time domain $[0, t_m]$. They are used to calculate the non-equilibrium probability density of the system $p(x, t)$ by the formula

$$p(x_j, t_i) = \frac{N_j}{N\Delta x} , \quad (4)$$

where N_j is the number of realizations, which are in the interval $[x_j, x_j + \Delta x)$ in the time moment t_i; Δx is the space sampling step. This formula can be also used to find the equilibrium probability density $p(x)$, which is determined for the time moment t_m. And the condition $t_m >> a^{-1}$, where a^{-1} is the system relaxation time, should hold.

4. RESULTS AND DISCUSSION

Numerical experiment is carried out for the following fixed parameters of the system: coefficient of linear restoring force, $a = 1$; intensity of additive noise, $\sigma_2^2 = 4$; cross-correlation coefficient, $r = 0{,}9$. Intensity of multiplicative noise is varied, i.e., this parameter is taken to be the control one. Its corresponding critical value is $\sigma_{1cr} = a/(r\sigma_2) \approx 0{,}56$. Parameters of the algorithm of the numerical experiment are: $\Delta t = 0{,}01$; $\Delta x = 0{,}02$; $t_m = 5$; $N = 10^8$ (if it is not specified differently in the caption).

In Fig. 1 we show plots of the equilibrium probability density $p(x)$ obtained according to the exact expression (2) and algorithm of the numerical experiment. At $\sigma_1 < \sigma_{1cr}$ $p(x)$ is the unimodal function with the global maximum in the point $x = 0$ (Fig. 1a), the most probable state of the system in stochastic dynamics corresponds to the stable point in deterministic dynamics. At $\sigma_1 > \sigma_{1cr}$ equilibrium probability density is the bimodal function with two local maxima of the same height and one local minimum at zero (Fig. 1b), the most probable states of the system do not correspond any more to the stable point in deterministic dynamics. Therefore, at $\sigma_1 = \sigma_{1cr}$ the unimodal-bimodal transition induced by the cross-correlated noises occurs.

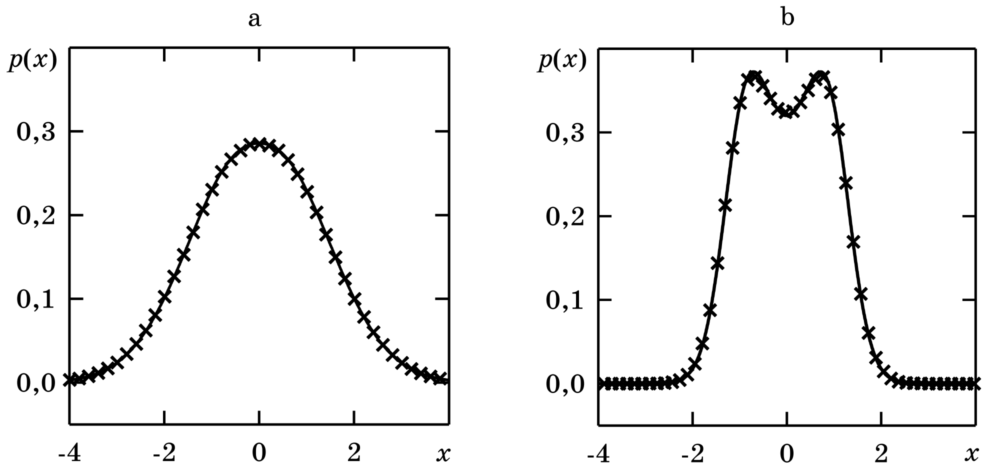

Fig. 1 *– Equilibrium probability density. Analytical results are represented by the firm line, numerical results – by the crosses.* $\sigma_1 = 0{,}2$ *(a);* $\sigma_1 = 2$ *(b)*

As seen from Fig. 1, numerical results agree well with the analytical ones both qualitatively and quantitatively. This implies the correctness of the proposed algorithm of the numerical experiment, and possibility of its further application for the calculation of the non-equilibrium probability density of the system, whose exact expression can not be obtained analytically.

In Fig. 2 we show the temporal evolution of the non-equilibrium probability density of the system to the equilibrium one for the case $\sigma_1 = 2$ ($\sigma_1 > \sigma_{1cr}$). At the initial time moment, probability density is the delta-peak at zero, $p(x) = \delta(x)$, i.e., it is the unimodal function. After some critical time t_{cr} it becomes a plane with the double maximum (curve 2, Fig. 2), then it is transformed into the bimodal function with two maxima, which are the most expressed for the equilibrium probability density (curve 4, Fig. 2).

Using values of the non-equilibrium probability density at different values of the control parameter σ_1, which are larger than σ_{1cr}, we find the critical time. For this purpose we plot the temporal evolution of the probability density maximum points, which determine the ordering parameter (Fig. 3). It is seen from this figure that when the control parameter tends from above to the critical one, the time of change of the initially unimodal probability density to the bimodal one increases, i.e., system remains longer and longer in the intermediate state $x_m = 0$. This implies the critical slowing down of the non-equilibrium transition dynamics.

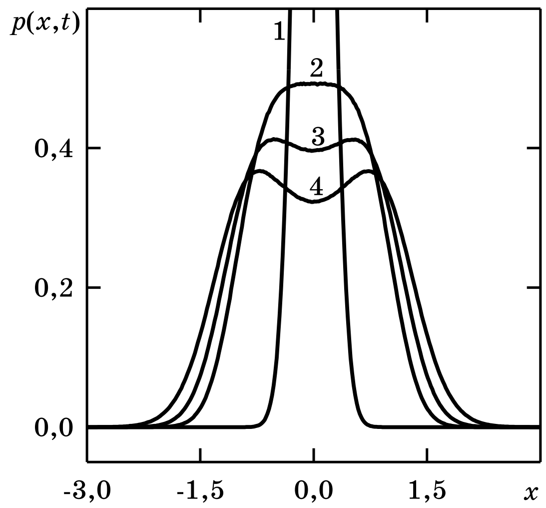

Fig. 2 *– Evolution of the non-equilibrium probability density.*
1 – t = 0,01*; 2 – t* = 0,21*; 3 – t* = 0,4*; 4 – t* = 5

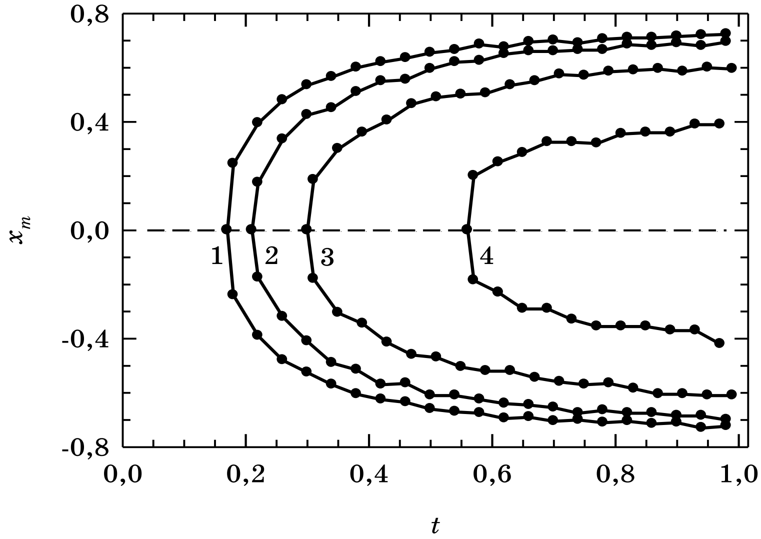

Fig. 3 *– Evolution of the maximum points. 1 –* σ_1 = 2,5*; 2 –* σ_1 = 2*;*
3 – σ_1 = 1,5*; 4 –* σ_1 = 1*. The number of sample trajectories is* $N = 10^9$

5. CONCLUSIONS

The numerical experiment for the dynamic system excited by two Gaussian white noises is performed. The values of the equilibrium probability density are obtained. It is established that the system demonstrates unimodal-bimodal transition induced by the cross-correlated noises in full accordance with the results obtained by exact analytical methods in Ref. [4]. The non-equilibrium probability density, which can not be calculated by exact methods, is found. The temporal evolution of its maximum points, which determine the ordering parameter of the transition, is studied. It is established that the time of change of the initially unimodal probability density to the bimodal one increases as the control parameter decreases to the critical value. This implies the critical slowing down of the non-equilibrium transition dynamics. This feature supplements the analogy of unimodal-bimodal transition in the considered system with classical phase transitions. The proposed algorithm of the numerical experiment can be used in the following when investigating the similar systems.

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