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VARIANCE OF THE COORDINATES OF LOCALIZATION FOR PARTICLES IN A RANDOM SAWTOOTH POTENTIAL

E.S. Denisova

Sumy State University,
2, Rimsky-Korsakov Str., 40007, Sumy, Ukraine
E-mail: es_denisova@mail.ru

A new regime of the directed transport of particles in a random sawtooth potential driven by an alternating external force is studied. For this transport regime, which is characterized by zero average velocity of particles and a finite transport distance, the variance of coordinates of particles localization is calculated exactly and analyzed for a particular case of the random sawtooth potential. It has been established that the variance plays an important role in this transport regime. In particular, the root-mean-square displacement of particles can essentially exceed their average displacement in the preferred direction.

Keywords*: RATCHET SYSTEMS, RANDOM SAWTOOTH POTENTIAL, DIRECTED TRANSPORT, VARIANCE OF COORDINATES, PARTICLES LOCALIZATION, ROOT-MEAN-SQUARE DISPLACEMENT.*

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1. INTRODUCTION

The study of ratchet systems, i.e., systems in which particles or other objects acquire directed motion under the action of undirected (random or periodic) forces, is of great interest. In particular, the ratchet-effect forms the theoretical basis of the so-called molecular motors, i.e., nanodevices, that transform thermal, chemical or any other energy into mechanical one [1-3].

Usually the action of ratchet systems on transported particles is modelled by spatially asymmetric periodic potentials, and the dynamics of these particles is described by the Langevin equation. The assumption of spatial periodicity of ratchet potentials essentially simplifies the analysis of the solutions of this equation, but actually it is difficult to realize in practice. The reason is that there always exist unremovable inhomogeneities in nanodevices, which violate a strict periodicity of model potentials. It is clear, therefore, that to study the transport properties of particles in such ratchet systems the random character of ratchet potentials should be taken into account. Within this approach it has been established, in particular, that inhomogeneities decrease the average velocity of particles [4, 5] and lead to the diffusion motion [6-8].

Recently, on the example of a random sawtooth potential we showed [9] that in disordered ratchet systems a new transport regime of particles can exist. In contrast to usual regime, which is characterized by nonzero average velocity of particles and an arbitrary large transport distance, this transport regime is characterized by zero average velocity of particles and a finite transport distance. In particular, we derived the average value of the maximum displacement of particles in the preferred transport direction. In this work we determine one more important characteristic of a new transport regime, namely, the variance of this displacement and analyze its dependence on the parameters of the model.

2. EQUATION OF MOTION

As in Ref. [9], we use the following dimensionless equation of particles motion:

$$\dot{X}_t = g(X_t) + f(t) \quad (X_0 = 0). \tag{1}$$

Here X_t denotes the particle coordinate at time t; $g(x) = -dU(x)/dx = \pm g_\pm$ represents the dichotomous random force generated by a random sawtooth potential $U(x)$ (see Fig. 1); $f(t)$ is a periodic external force of a period $2T$. We assume, that the random potential $U(x)$, describing the influence of ratchet system, is characterized by (i) statistically independent random intervals s_j, distributed with the probability densities $p_+(s)$ and $p_-(s)$ for even numbered ($j = 2n$, $n = 0, \pm 1, \ldots$) and odd numbered ($j = 2n + 1$) intervals, respectively; (ii) two slopes $-g_+$ and g_- ($g_+ > g_- > 0$); and (iii) the condition $g_+ s_+ = g_- s_-$, where $s_\pm = \int_0^\infty ds s p_\pm(s)$ are the average lengths of even (s_+) and odd (s_-) intervals. In compliance with the last condition the average force $\lim_{L\to\infty}(1/2L)\int_{-L}^{L} dx g(x)$ acting on a particles in this potential equals zero.

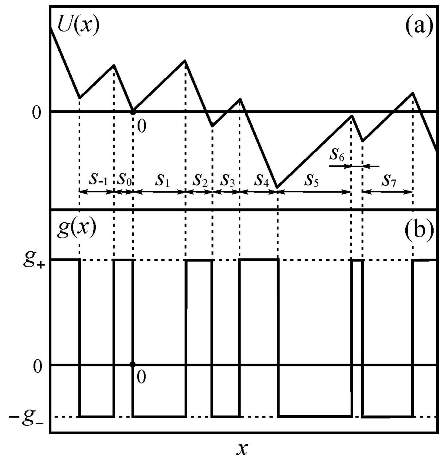

***Fig. 1** – Schematic representation of one realization of the random sawtooth potential (a) and the corresponding realization of the random dichotomous force (b)*

Periodic force $f(t)$ is assumed to be alternating, i.e., $f(t) = (-1)^{k+1} f$, where f (> 0) is the force amplitude; $k = [t/T] + 1$; and $[t/T]$ is the integer part of t/T. As in the case of the random dichotomous force $g(x)$, the average value of the force $f(t)$ equals zero as well: $\lim_{\tau\to\infty}(1/\tau)\int_0^\tau dt f(t) = 0$. Although the average values of the forces are zero, in the present system the particle transport both with nonzero and zero average velocity can exist [9].

3. TRANSPORT WITH ZERO AVERAGE VELOCITY

The average transport velocity of particles is defined by the following formula:

$$\upsilon_T = \lim_{t\to\infty} \frac{\langle X_t \rangle}{t}, \tag{2}$$

where the angular brackets denote an averaging over all realizations of a random sawtooth potential. According to this definition, the unlimited growth of $\langle X_t \rangle$ ($\langle X_t \rangle > 0$ if $g_+ > g_-$) at $t \to \infty$ is the necessary condition of existence of nonzero particles velocity. The transport regime, we have interested with, is characterized by finite value of $\langle X_\infty \rangle$, and, as a result, by zero average velocity $\upsilon_T = 0$. As it was shown in [9], this regime takes place in those cases only when the probability $\int_\Delta^\infty ds\, p_-(s)$ that the length of the intervals s_{2n+1} exceeds the distance $\Delta = (f - g_-)T$ is not zero. In other words, the average velocity equals zero if realizations of a random dichotomous force $g(x)$ contain so long intervals s_{2n+1} that particles cannot overcome them in a positive direction of the axis x during the half-period T.

According to the above-mentioned statement, the distance

$$l_{2n} = \sum_{j=1}^{2n} s_j \tag{3}$$

from the origin of the coordinates to the first "impassable" interval s_{2n+1} ($n = 0, 1, \ldots$), located in the positive part of the axis x, is of great interest for the description of the features of this transport regime. If $f \in (g_-, g_+)$ then a particle, initially placed in the origin of the coordinates, after some time will certainly be in the neighborhood of a point $x = l_{2n}$, where it will oscillate. If $f > g_+$ then particle can get the vicinity of this point with some probability only, since it can move with finite probability in the negative part of the axis x.

Since the function $f(x)$ is random, the distances l_{2n} are random as well. Therefore the main quantity characterizing these distances is their probability density $P(l)$, which can be written as follows:

$$P(l) = (1-w)\sum_{n=1}^{\infty} \int_0^\Delta \ldots \int_0^\Delta \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^\infty \ldots \int_0^\infty \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \delta(l - l_{2n}) + (1-w)\delta(l), \tag{4}$$

where $\delta(\cdot)$ is the Dirac delta-function; $w = \int_0^\Delta ds\, p_-(s)$ is the probability that the length of odd intervals does not exceed Δ. The most important characteristics of the random variable l_{2n} are its average value $\langle l \rangle$ and variance σ^2. For the average value, determining as $\langle l \rangle = \int_0^\infty dl\, l P(l)$, in general case of arbitrary probability densities $p_+(s)$ and $p_-(s)$ we obtained a simple formula

$$\langle l \rangle = \frac{\tilde{s}_- + s_+ w}{1 - w}, \tag{5}$$

where

$$\tilde{s}_- = \int_0^{\Delta} dssp_-(s), \quad s_+ = \int_0^{\infty} dssp_+(s)\,. \tag{6}$$

Analysis of the expression (5) showed, in particular, that the transition from one transport regime to another can occur under certain conditions. As for the variance, which is a measure of the spreading of particle localization, it was not studied before.

4. VARIANCE OF THE COORDINATES OF PARTICLES LOCALIZATION

According to the definition, the variance is given by $\sigma^2 = \langle l^2 \rangle - \langle l \rangle^2$, where

$$\langle l^2 \rangle = \int_0^{\infty} dl\, l^2 P(l)\,. \tag{7}$$

Substituting the probability density (4) into (7) and using the delta-function properties, we obtain:

$$\langle l^2 \rangle = (1-w) \sum_{n=1}^{\infty} \int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) l_{2n}^2\,. \tag{8}$$

For simplifying the further calculations we use the formula

$$l_{2n}^2 = \sum_{m=1}^{n} s_{2m-1}^2 + \sum_{m=1}^{n} s_{2m-1} \sum_{\substack{k=1\\k\neq m}}^{n} s_{2k-1} + \sum_{m=1}^{n} s_{2m}^2 + \sum_{m=1}^{n} s_{2m} \sum_{\substack{k=1\\k\neq m}}^{n} s_{2k} + 2\sum_{m=1}^{n} s_{2m-1} \sum_{k=1}^{n} s_{2k}\,, \tag{9}$$

which follows from definition (3). Integration of each term gives:

$$\begin{aligned}
&\int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^{n} s_{2m-1}^2 = n\tilde{q}_- w^{n-1}\,,\\
&\int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^{n} s_{2m-1} \sum_{\substack{k=1\\k\neq m}}^{n} s_{2k-1} = n(n-1)\tilde{s}_-^2 w^{n-2}\,,\\
&\int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^{n} s_{2m}^2 = nq_+ w^{n}\,, \qquad (10)\\
&\int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^{n} s_{2m} \sum_{\substack{k=1\\k\neq m}}^{n} s_{2k} = n(n-1)s_+^2 w^{n}\,,\\
&\int_0^{\Delta} \ldots \int_0^{\Delta} \left(\prod_{j=1}^{n} ds_{2j-1} p_-(s_{2j-1}) \right) \int_0^{\infty} \ldots \int_0^{\infty} \left(\prod_{k=1}^{n} ds_{2k} p_+(s_{2k}) \right) \sum_{m=1}^{n} s_{2m-1} \sum_{k=1}^{n} s_{2k} = n^2 \tilde{s}_- s_+ w^{n-1}\,,
\end{aligned}$$

where

$$\tilde{q}_- = \int_0^{\Delta} dss^2 p_-(s), \quad q_+ = \int_0^{\infty} dss^2 p_+(s)\,. \tag{11}$$

Thus, in accordance with the above results, formula (8) becomes

$$\langle l^2\rangle = (1-w)\sum_{n=1}^{\infty}\left[n\tilde{q}_-w^{n-1} + n(n-1)\tilde{s}_-^2w^{n-2} + nq_+w^n + n(n-1)s_+^2w^n + 2n^2\tilde{s}_-s_+w^{n-1}\right]. \quad (12)$$

Series in (12) can be easily calculated using the geometric series formula $\sum_{n=0}^{\infty} w^n = (1-w)^{-1}$. This yields

$$\langle l^2\rangle = \frac{\tilde{q}_- + q_+w + 2\tilde{s}_-s_+}{1-w} + 2\left(\frac{\tilde{s}_- + s_+w}{1-w}\right)^2 . \quad (13)$$

Then taking into account expression (5) for the average particles velocity and definition of the variance of coordinates of particles localization $\sigma^2 = \langle l^2\rangle - \langle l\rangle^2$, we obtain

$$\sigma^2 = \frac{\tilde{q}_- + q_+w + 2\tilde{s}_-s_+}{1-w} + \langle l^2\rangle . \quad (14)$$

As an application example, we consider the exponential distribution of intervals s_j, assuming that $p_\pm(s) = \lambda_\pm e^{-\lambda_\pm s}$, where λ_+ and λ_- are the positive distribution parameters for even and odd intervals, respectively. According to definitions (6) and (11), in this case

$$s_\pm = \frac{1}{\lambda_\pm},\ q_+ = \frac{2}{\lambda_+^2},\ \tilde{q}_- = 2\frac{\tilde{s}_-}{\lambda_-} - \Delta^2 e^{-\lambda_-\Delta},\ \tilde{s}_- = \frac{1}{\lambda_-}\left(1 - e^{-\lambda_-\Delta} - \lambda_-\Delta e^{-\lambda_-\Delta}\right), \quad (15)$$

and the condition that the average value of the random dichotomous force equals zero, $g_+s_+ = g_-s_-$, allows to exclude from consideration, for example, parameter λ_+: $\lambda_+ = \lambda_+g_+/g_-$. Finally, taking into account that $w = 1 - e^{-\lambda_-\Delta}$, from (5) and (14) we find

$$\langle l\rangle = \frac{1}{\lambda_-}\left(1 + \frac{g_-}{g_+}\right)\left(e^{\lambda_-\Delta} - 1\right) - \Delta \quad (16)$$

and

$$\sigma^2 = \frac{2}{\lambda_-^2}\left(1 + \frac{g_-}{g_+} + \frac{g_-^2}{g_+^2}\right)\left(e^{\lambda_-\Delta} - 1\right) - \frac{2}{\lambda_-}\left(1 + \frac{g_-}{g_+}\right)\Delta - \Delta^2 + \langle l^2\rangle . \quad (17)$$

In particular, at $\Delta \to 0$ and $\Delta \to \infty$ these formulas reduce, respectively, to

$$\langle l\rangle = \frac{g_-}{g_+}\Delta, \quad \sigma^2 = \frac{2}{\lambda_-}\frac{g_-^2}{g_+^2}\Delta \quad (18)$$

and

$$\langle l\rangle = \frac{1}{\lambda_-}\left(1 + \frac{g_-}{g_+}\right)e^{\lambda_-\Delta}, \quad \sigma^2 = \langle l\rangle^2 . \quad (19)$$

The considered example shows that in random ratchet systems the variance of the coordinates of particles localization plays the important role both at $\Delta \ll 1$ (the case of small amplitude values and/or external force period) and at $\Delta \gg 1$. According to (19), in the last case the root-mean-square value of the coordinates of particles localization σ has the same order as the average value $\langle l \rangle$, i.e., $\sigma \sim \langle l \rangle$. In the first case the variance plays a more important role since $\sigma \gg l$, as it follows from (18).

5. CONCLUSIONS

A new regime of particles transport induced by an alternating external force in a random sawtooth potential is studied. For this transport regime, which is characterized by zero average velocity of particles and their finite displacement in a preferential direction, the explicit formula for the variance of the coordinates of particles localization is found and its analysis in particular case of a sawtooth potential is carried out. It is established that the variance is an important characteristic of the given transport regime. Particularly, the root-mean-square value of the coordinates of particles localization can essentially exceed the average distance of their displacement in a preferential direction.

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