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New Higher Excited Energy Levels of Rydberg States for Weakest Bound Potential Model Theory: Extended Quantum Mechanics

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In present research paper, a Bopp's shift method and standard perturbation theory are used to find exact analytical solutions of the noncommutative 3-dimensional space phase (NC: 3D-RSP) for modified time-independent Schrödinger equation of weakest bound electron potential model (WBEPM) theory for neutral indium. We have obtained the explicit higher energy eigenvalues for n^{th} excited states. Furthermore, the obtained corrections of energies are depended on the discreet atomic quantum numbers ($j = l \pm 1/2$, $\left(n^*, l^*\right) \equiv (n+d, l+d)$, $S_{\overline{SL}}$ and m), in addition to the four infinitesimal parameters (Θ, χ) and $\left(\bar{\theta}, \bar{\sigma}\right)$ which are induced by position-position and momentum-momentum noncommutativity, respectively. We have also shown that, the total complete degeneracy of higher energy level of the modified (WBEPM) theory equal the new values $2n^2$.

Keywords: The weakest bound potential, Noncommutative space and phase, Star product and Bopp's shift method.

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1. INTRODUCTION

Over the past ten years, Zheng et al. have developed the (WBEPM) theory for many electron systems and obtained very satisfactory results with theoretical and experimental results in the literature for calculation of spectroscopic data [1-5]. The study of weakest bound potential has now become a very interest field due to their applications in different fields [6]. The noncommutative symmetries which was introduced firstly by H. Snyder [7], considered as a logical extended of ordinary quantum mechanics, have been the subject of studies in recent times, we want to extended, this study to case of noncommutative space phase to obtaining an profound new interpretations in the subatomics scales on based to our previously works [8-13] and other works in this context [14-20]. The nonrelativistic energy levels for neutral indium in the context of noncommutative space have not been obtained yet. This is the priority for this work. The modified (WBEPM) theory used in this framework takes the form (see below):

$$V_{\text{nc-ni}}\left(\hat{r}_i\right) = -\frac{Z^*}{r_i} + \frac{\left[d(d+1)+2dl\right]}{r_i^2} + \left(\frac{\left[d(d+1)+2dl\right]}{r_i^4} - \frac{Z^*}{2r_i^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\bar{\vec{\theta}}}{2} \quad (1)$$

The crucial purpose of this paper is to determine the energy levels of above potential in (NC: 3D-RSP) symmetries using the generalization Bopp's shift method which depend on the concepts that we present below and in the third section to discover the new symmetries and a possibility to obtain another applications to this potential in different fields. The new structure of extended quantum mechanics based to new noncommutative canonical commutations relations (NNCCRs) in both Schrödinger and Heisenberg pictures ((SP) and (HP)), respectively, as follows (Throughout this paper the natural units $c = \hbar = 1$ will be used) [9, 10, 13-16]:

$$\begin{cases} \left[x_i, p_j\right] = i\delta_{ij} \\ \left[x_i, x_j\right] = 0 \\ \left[p_i, p_j\right] = 0 \end{cases} \Rightarrow \begin{cases} \left[\hat{x}_i(t) \overset{*}{,} \hat{p}_j(t)\right] = i\delta_{ij} \\ \left[\hat{x}_i \overset{*}{,} \hat{x}_j\right] = \left[\hat{x}_i(t) \overset{*}{,} \hat{x}_j(t)\right] = i\theta_{ij} \\ \left[\hat{p}_i \overset{*}{,} \hat{p}_j\right] = \left[\hat{p}_i(t) \overset{*}{,} \hat{p}_j(t)\right] = i\bar{\theta}_{ij} \end{cases} \quad (2)$$

However, the new operators $\left(\hat{x}_i(t), \hat{p}_i(t)\right)$ in (HP) are depending to the corresponding new operators $\left(\hat{x}_i, \hat{p}_i\right)$ in (SP) from the following projections relations, respectively [11]:

$$\begin{aligned} &\left(x_i(t), p_i(t)\right) = \exp(i\hat{H}_{ni}(t-t_0))\left(x_i, p_i\right)\exp(-i\hat{H}_{ni}(t-t_0)) \Rightarrow \\ &\left(\hat{x}_i(t), \hat{p}_i(t)\right) = \exp(i\hat{H}_{nc-ni}(t-t_0)) * \left(\hat{x}_i, \hat{p}_i\right) * \exp(-i\hat{H}_{nc-ni}(t-t_0)) \end{aligned} \quad (3)$$

While the dynamics of new systems ($\frac{d\hat{x}_i(t)}{dt}$, $\frac{d\hat{p}_i(t)}{dt}$) are described from the following relations:

$$\begin{aligned} &\frac{dx_i(t)}{dt} = \left[\hat{x}_i(t), \hat{H}_{ni}\right] \Rightarrow \frac{d\hat{x}_i(t)}{dt} = \left[\hat{x}_i(t) \overset{*}{,} \hat{H}_{nc-ni}\right] \\ &\frac{dp_i(t)}{dt} = \left[p_i(t), \hat{H}_{ni}\right] \Rightarrow \frac{d\hat{p}_i(t)}{dt} = \left[\hat{p}_i(t) \overset{*}{,} \hat{H}_{nc-ni}\right] \end{aligned} \quad (4)$$

here $\hat{H}_{ni}$ and $\hat{H}_{nc-ni}$ denote to the ordinary and new quantum Hamiltonian operators in the quantum me-

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chanics and it's extension while $\frac{\mathrm{d}\hat{x}_i(t)}{\mathrm{dt}}$ and $\frac{\mathrm{d}\hat{p}_i(t)}{\mathrm{dt}}$ are describe the dynamics of systems in (NC: 3D-RSP). The very small two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes and $(*)$ denote to the new star product, which is generalized between two arbitrary functions (f,g) $(x,p)\rightarrow(\hat{f},\hat{g})(\hat{x},\hat{p})$ to $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})\equiv(f*g)(x,p)$ instead of the usual product $(fg)(x,p)$ in ordinary 3-dimensional spaces [12, 15-19]:

$$(fg)(x,p)\Rightarrow(f*g)(x,p)$$
$$=\left(fg-\frac{i}{2}\left(\theta^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu}+\bar{\theta}^{\mu\nu}\frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial p^\nu}\right)\right)(x,p)\Big|_{(x^\mu=x^\nu,p^\mu=p^\nu)}+O\left(\theta^2,\bar{\theta}^2\right) \quad (5)$$

the second and the third terms are induced by (space-space) and (phase-phase) noncommutativity properties, respectively. The organization scheme of the study is given as follows: In next section, we briefly review the Schrödinger equation with ordinary potential $V(r_i)$ on based to refs. [4, 6]. The Section 3, devoted to studying the 3-dimensional modified Schrödinger equation by applying Bopp's shift method for (WBEPM) theory. In the fourth section and by applying standard perturbation theory we find the quantum spectrum of the higher n^{th} excited levels for spin-orbital interaction in the framework of the (NC-3D: RSP) symmetries. In the next section, we derive the magnetic spectrum for studied potential $V_{\text{nc-ni}}(\hat{r}_i)$. In the sixth section, we resume the global spectrum and corresponding noncommutative Hamiltonian operator for (WBEPM) theory and corresponding energy levels. Finally, we give a brief conclusion in last section.

2. REVIEW OF THE EIGNENFUNCTIONS AND THE ENERGY EIGENVALUES FOR NEUTRAL INDIUM IN ORDINARY THREE DIMENSIONAL SPACES

We shall recall here the time independent Schrödinger equation for a weakest bound potential, which can be divided into two parts, the first one is the Coulomb potential $\left(-\frac{Z^*}{r_i}\right)$ and the other one $\frac{[d(d+1)+2dl]}{r_i^2}$ is the dipole potential [4, 6]:

$$V(r_i)=-\frac{Z^*}{r_i}+\frac{[d(d+1)+2dl]}{r_i^2} \quad (6)$$

where Z^* and r_i are the effective nuclear charge and the distance between the weakest bound electron and nucleus, respectively. The terms containing eq. (6) have positive and negative contributions. If we insert this potential into the Schrödinger equation:

$$\left(-\frac{\Delta}{2}-\frac{Z^*}{r_i}+\frac{d(d+1)+2dl}{r_i^2}\right)\Psi(\vec{r})=E\Psi(\vec{r}) \quad (7)$$

The electronic radial wave functions are shown as a function of the Laguerre polynomial in terms of some parameters [4, 6]:

$$R(r_i)=\left(\frac{2Z^*}{n^*}\right)^{l^*+3/2}\left[\frac{2n^*}{(n^*-l^*-1)!}\Gamma(n^*-l^*+1)\right]^{-1/2}$$
$$\exp\left(-\frac{Z^*}{n^*}r_i\right)r^{l^*}L_{n^*-l^*-1}^{2l^*+1}\left(\frac{2Z^*}{n^*}r_i\right) \quad (8)$$

where $(n^*,l^*)=(n+d,l+d)$ are defined to be effective principal quantum number, and effective azimuthal quantum number, respectively. In spherical coordinates, the complete wave function $\Psi(r,\theta,\varphi)$ and the energy eigenvalues are given by [4, 6]:

$$\Psi(r,\theta,\varphi)=\left(\frac{2Z^*}{n^*}\right)^{l^*+3/2}\left[\frac{2n^*}{(n^*-l^*-1)!}\Gamma(n^*-l^*+1)\right]^{-1/2}$$
$$\exp\left(-\frac{Z^*}{n^*}r_i\right)r^{l^*}L_{n^*-l^*-1}^{2l^*+1}\left(\frac{2Z^*}{n^*}r_i\right)Y_l^m(\theta,\varphi) \quad (9)$$
$$\varepsilon=-\frac{(Z^*)^2}{2n^{*2}}$$

Where $n^*=n-\delta$ and δ given by [6]:

$$\delta=a+b(n-\delta_0)^{-2}+c(n-\delta_0)^{-4}+d(n-\delta_0)^{-6} \quad (10)$$

which present defect for a given fixed orbital quantum number while a, b, c and d are coefficients.

3. THREE DIMENSIONAL NONCOMMUTATIVE REAL SPACE-PHASE FOR (WBEPM) THEORY

In this section, we shall gives an overview for the weakest bound potential $V(r_i)$ in (NC: 3D-RSP), to perform this task the physical form of Schrödinger equation should be written as [11, 12]:

Ordinary three dimensional Hamiltonian operators $\hat{H}(p_i,x_i)$ will be replace by new three Hamiltonian operators $\hat{H}_{nc-ni}(\hat{p}_i,\hat{x}_i)$,

Ordinary complex wave function $\Psi(\vec{r})$ will be replacing by new complex wave function $\hat{\Psi}(\vec{\hat{r}})$.

Ordinary energy E will be replaced by new values E_{nc-ni}.

And the last step corresponds to replace the ordinary old product by new star product $(*)$, which allow us to constructing the modified Schrödinger equations in both (NC-3D: RSP) as:

$$\hat{H}_{nc-ni}(\hat{p}_i, \hat{x}_i) * \hat{\Psi}(\vec{r}) = E_{nc-ni}\hat{\Psi}(\vec{r}) \quad (11)$$

Instead of solving any quantum systems by using directly star product procedure, a Bopp's shift method can be used [16-20]:

$$[\hat{x}_i, \hat{x}_j] = [\hat{x}_i(t), \hat{x}_j(t)] = i\theta_{ij} \text{ and } [\hat{p}_i, \hat{p}_j] = [\hat{p}_i(t), \hat{p}_j(t)] = i\bar{\theta}_{ij} \quad (12)$$

The new generalized positions and momentum coordinates $(\hat{x}_i, \hat{p}_i)$ in (NC: 3D-RSP) are depended with corresponding usual generalized positions and momentum coordinates (x_{ii}, p_i) in ordinary quantum mechanics by the following, respectively [14-19]:

$$\begin{cases} \hat{x}_i = x_i - \frac{\theta_{ij}}{2} p_j \\ \hat{p}_i = p_i - \frac{\bar{\theta}_{ij}}{2} x_j \end{cases} \quad (13)$$

Which allow us to getting the two operators ($\hat{r}^2$ and $\hat{p}^2$) in (NC-3D: RSP), respectively [10, 11]:

$$\hat{r}_i^2 = r_i^2 - \vec{\mathbf{L}}\vec{\Theta} \quad \text{and} \quad \hat{p}_i^2 = p_i^2 + \vec{\mathbf{L}}\bar{\vec{\boldsymbol{\theta}}} \quad (14)$$

Where the two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\bar{\vec{\boldsymbol{\theta}}}$ are $(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13})$ and $(L_x\bar{\theta}_{12} + L_y\bar{\theta}_{23} + L_z\bar{\theta}_{13})$, respectively and ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ and $\Theta_{ij} = \theta ij/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$H(\hat{p}_i, \hat{x}_i)\psi(\vec{r}) = E_{nc-ni}\psi(\vec{r}) \quad (15)$$

Where the new operator of Hamiltonian $H(\hat{p}_i, \hat{x}_i)$ can be expressed as:

$$H_{nc-ni}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{x}_i = x_i - \frac{\theta_{ij}}{2} p_j \text{ and } \hat{p}_i = p_i - \frac{\bar{\theta}_{ij}}{2} x_j\right) \quad (16)$$

After straightforward calculations, we can obtain the two important terms, which will be use to determine the modified weakest bound potential in (NC: 3D-RSP):

$$\begin{aligned} &[d(d+1)+2dl]\hat{r}_i^{-2} = [d(d+1)+2dl]r_i^{-2} + \frac{[d(d+1)+2dl]}{r_i^4}\vec{\mathbf{L}}\vec{\Theta}, \\ &-Z^*\hat{r}_i^{-1} = Z^* r_i^{-1} - \frac{Z^*\vec{\mathbf{L}}\vec{\Theta}}{2r_i^3} \end{aligned} \quad (17)$$

From above relations, one can write the deformed operator $V_{\text{ni}}(\hat{r})$ for modified weakest bound potential and the noncommutative kinetic term $\frac{\hat{p}^2}{2}$, respectively:

$$\begin{aligned} &V_{\text{ni}}(\hat{r}_i) = -\frac{Z^*}{\hat{r}_i} + \frac{[d(d+1)+2dl]}{\hat{r}_i^2} \\ &\frac{\hat{p}^2}{2} = \frac{\vec{p}^2}{2} + \frac{\vec{\mathbf{L}}\bar{\vec{\boldsymbol{\theta}}}}{2} \end{aligned} \quad (18)$$

Which allow us to obtaining the global potential operator $H_{\text{nc-ni}}(\hat{r})$ in (NC: 3D-RSP) as:

$$\begin{aligned} H_{\text{nc-ni}}(\hat{r}) = &-\frac{Z^*}{r_i} + \frac{[d(d+1)+2dl]}{r_i^2} + \\ &\left(\frac{[d(d+1)+2dl]}{r_i^4} - \frac{Z^*}{2r_i^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\bar{\vec{\boldsymbol{\theta}}}}{2} \end{aligned} \quad (19)$$

It's clearly, that the two first terms are given the ordinary weakest bound potential in three dimensional space, while the rest terms are proportional's with two infinitesimals parameters (Θ and $\bar{\theta}$) and then gives the terms of perturbations $H_{\text{per-ni}}(r)$ in (NC: 3D-RSP) as:

$$H_{\text{per-ni}}(r) = \left(\frac{[d(d+1)+2dl]}{r_i^4} - \frac{Z^*}{2r_i^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\bar{\vec{\boldsymbol{\theta}}}}{2} \quad (20)$$

4. THE EXACT SPIN-ORBITAL SPECTRUM MODIFICATIONS FOR MODIFIED (WBEPM) THEORY IN (NC:3D- RSP):

Again, the perturbative term $H_{\text{per-ni}}(r)$ can be rewritten to the equivalent physical form:

$$H_{\text{per-ni}}(r) = 2\left(\Theta\left(\frac{[d(d+1)+2dl]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\theta}}{2}\right)\vec{S}\vec{L} \quad (21)$$

Furthermore, the above perturbative terms $H_{\text{per-ni}}(r)$ can be rewritten to the following new form:

$$H_{\text{per-ni}}(r) = \left(\Theta\left(\frac{[d(d+1)+2dl]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\theta}}{2\mu}\right)\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \quad (22)$$

We just replace $\vec{S}\vec{L}$ by the expression $\frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$, in quantum mechanics, this operator traduces the coupling between spin and orbital momentum. The set ($\hat{H}_{\text{so-ni}}(r)$, J^2, L^2, S^2 and J_z) forms a complete of conserved physics quantities and the eigen-values of the spin orbital coupling operator are $k_\pm \equiv \frac{1}{2}\left\{\left(l \pm \frac{1}{2}\right)(l \pm \frac{1}{2} + 1) + l(l+1) - \frac{3}{4}\right\}$ corresponding: $j = l + \frac{1}{2}$ (spin up) and $j = l - \frac{1}{2}$ (spin down), respectively, then, one can form a diagonal (3×3) matrix, with non null elements are $(\hat{H}_{so-ni})_{11}$, $(\hat{H}_{so-ni})_{22}$ and $(\hat{H}_{so-ni})_{33}$ for modified potential in (NC: 3D-RSP) as:

$$\left(\hat{H}_{so-ni}\right)_{11} = k_{+}\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\theta}}{2m_0}\right)$$

if $j = l+\frac{1}{2}$ $\Rightarrow$ spin -up

$$\left(\hat{H}_{so-ni}\right)_{22} = k_{-}\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\theta}}{2m_0}\right) \quad (23)$$

if $j = l-\frac{1}{2}$ $\Rightarrow$ spin -down

$$\left(\hat{H}_{so-ni}\right)_{33} = 0$$

For, the non-weakest bound electrons (the Schrödinger equation in multi electron systems), $\overline{SL}$ coupling is the dominant coupling scheme in light atoms and electric dipole line strength for transitions between two excited levels in this coupling scheme can be given to be [4, 5, 21]:

$$\begin{aligned}\sqrt{S_{\overline{SL}}} &\equiv \left\langle\left[\left(...a_1L_1,l_2\right)L\left(...S_1s_2\right)J\left|r_N^{(1)}\right|*\left(...a'_1L'_1,l'_2\right)L'\left(...S'_1s'_2\right)J'\right]\right\rangle \\ &= (-)^{S+J'+L_1+l'_2}\left[J,J',L,L'\right]^{1/2} * \begin{Bmatrix} L & S & J \\ J' & 1 & L' \end{Bmatrix}\begin{Bmatrix} L_1 & l_2 & L \\ 1 & L' & l'_2 \end{Bmatrix} P^{(1)}_{l_2 l'_2}\end{aligned} \quad (24)$$

With $P^{(1)}_{l_2l'_2} = \int_0^{+\infty} r^{k+2}R_{n_il_i}(r)R_{n_jl_j}(r)dr$ is the transition matrix element and $\begin{Bmatrix} L & S & J \\ J' & 1 & L' \end{Bmatrix} = W(abcd;ef)$ which known by Racah coefficient or Winger's 6-j symbol.

4.1 The Exact Spin-orbital Spectrum Modifications for Modified (WBEPM) Theory in (*NC: 3D- RSP*)

In order to obtain the bound solutions at higher n^{th} excited levels, we first the find the corrections $E_{\text{u-ni}}$ and $E_{\text{d-ni}}$ for spin up and spin down, respectively, at first order of two parameters Θ and $\bar{\theta}$ obtained by applying the standard perturbation theory:

$$\begin{aligned}E_{\text{u-ni}} &= \left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1} k_{+} \\ &\int_0^{+\infty}\exp\left(-2\frac{Z^*}{n^*}r_i\right)r^{2l^*}\left[L^{2l^*+1}_{n^*-l^*-1}\left(\frac{2Z^*}{n^*}r_i\right)\right]^2 \\ &\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\theta}}{2}\right)r_i^2dr_i\end{aligned} \quad (25)$$

$$\begin{aligned}E_{\text{d-ni}} &= \left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1} k_{-} \\ &\int_0^{+\infty}\exp\left(-2\frac{Z^*}{n^*}r_i\right)r^{2l^*}\left[L^{2l^*+1}_{n^*-l^*-1}\left(\frac{2Z^*}{n^*}r_i\right)\right]^2 \\ &\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\theta}}{2}\right)r_i^2dr_i\end{aligned} \quad (26)$$

A straightforward calculation yields:

$$E_{\text{u-ni}} = \left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1} k_{+}\left\{\Theta\left(T_1+T_2\right)+\frac{\bar{\theta}}{2}T_3\right\} \quad (27)$$

$$E_{\text{d-ni}} = \left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1} k_{-}\left\{\Theta\left(T_1+T_2\right)+\frac{\bar{\theta}}{2}T_3\right\} \quad (28)$$

Where, the three terms $T_i\left(Z^*,n^*,l^*\right)\left(i=\overline{1,3}\right)$ are given by:

$$\begin{aligned}T_1\left(Z^*,n^*,l^*\right) &= \left[d(d+1)+2dl\right]\int_0^{+\infty}\exp\left(-2\frac{Z^*}{n^*}r_i\right)r^{2l^*-1-1}\left[L^{2l^*+1}_{n^*-l^*-1}\left(\frac{2Z^*}{n^*}r_i\right)\right]^2 dr_i \\ T_3\left(Z^*,n^*,l^*\right) &= -Z^*\int_0^{+\infty}\exp\left(-2\frac{Z^*}{n^*}r_i\right)r^{2l^*-1}\left[L^{2l^*+1}_{n^*-l^*-1}\left(\frac{2Z^*}{n^*}r_i\right)\right]^2 dr_i \\ T_3\left(Z^*,n^*,l^*\right) &= \int_0^{+\infty}\exp\left(-2\frac{Z^*}{n^*}r_i\right)r^{2l^*+3-1}\left[L^{2l^*+1}_{n^*-l^*-1}\left(\frac{2Z^*}{n^*}r_i\right)\right]^2 dr_i\end{aligned} \quad (29)$$

We apply the following special integration [22]:

$$\begin{aligned}&\int_0^{+\infty} t^{\alpha-1}\exp(-\omega t)L_m^{\lambda}(\omega t)L_n^{\beta}(\omega t)dt = \\ &= \frac{\omega^{-\alpha}\Gamma(n-\alpha+\beta+1)\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\alpha+\beta)\Gamma(1+\lambda)} \\ &{}_3F_2(-m,\alpha,\alpha-\beta;-n+\alpha,\lambda+1;1)\end{aligned} \quad (30)$$

where ${}_3F_2(-m,\alpha,\alpha-\beta;-n+\alpha,\lambda+1;1)$ obtained from the generalized the hypergeometric function ${}_pF_q\left(\alpha_1,...,\alpha_p,\beta_1,....,\beta_q,z\right)$ for $p=3$ and $q=2$ while $\Gamma(x)$ denote to the usual Gamma function. After straightforward calculations, we can obtain the explicitly results:

$$\begin{aligned}T_1\left(Z^*,n^*,l^*\right) &= \left[d(d+1)+2dl\right]\frac{\left(2\frac{Z^*}{n^*}\right)^{-2l^*+1}\Gamma\left(n^*-l^*+2\right)\Gamma\left(n^*+l^*+1\right)}{\left(\left(n^*-l^*-1\right)!\right)^2\Gamma(3)\Gamma\left(2+2l^*\right)} \\ &{}_3F_2\left(-n^*+l^*+1,2l^*-1,-2;-n+2l^*-1,2l^*+2;1\right)\end{aligned} \quad (31)$$

$$\begin{aligned}T_2\left(Z^*,n^*,l^*\right) &= -Z^*\frac{\left(2\frac{Z^*}{n^*}\right)^{-2l^*}\Gamma\left(n^*-l^*+1\right)\Gamma\left(n^*+l^*+1\right)}{\left(\left(n^*-l^*-1\right)!\right)^2\Gamma(2)\Gamma\left(2+2l^*\right)} \\ &{}_3F_2\left(-n^*+l^*+1,2l^*,-1;-n^*+3l^*+1,2l^*+2;1\right)\end{aligned} \quad (32)$$

$$\begin{aligned}T_3\left(Z^*,n^*,l^*\right) &= \frac{\left(2\frac{Z^*}{n^*}\right)^{-2l^*-3}\Gamma\left(n^*-l^*-2\right)\Gamma\left(n^*+l^*+1\right)}{\left(\left(n^*-l^*-1\right)!\right)^2\Gamma(-3)\Gamma\left(2+2l^*\right)} \\ &{}_3F_2\left(-n^*+l^*+1,2l^*+3,2;-n^*+3l^*+4,2l^*+2;1\right)\end{aligned} \quad (33)$$

Which allow us to obtaining the exact modifications of n^{th} excited states $E_{\text{u-ni}}$ and $E_{\text{d-ni}}$ produced by spin-orbital effect:

$$E_{\text{u-ni}}\left(j,Z^*,n^*,l^*\right)=\left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}k_+\left\{\Theta T_{nc-ni}+\frac{\bar{\theta}}{2}T_3\right\} \quad (34)$$

$$E_{\text{d-ni}}\left(j,Z^*,n^*,l^*\right)=\left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}k_-\left\{\Theta T_{nc-ni}+\frac{\bar{\theta}}{2}T_3\right\} \quad (35)$$

we have introduced new parameters $T_{nc-ni}\left(Z^*,n^*,l^*\right)$ for the sake of simplicity:

$$\begin{aligned}T_{nc-ni}\left(Z^*,n^*,l^*\right)\equiv&\left[d(d+1)+2dl\right]\frac{\left(2\frac{Z^*}{n^*}\right)^{-2l^*+1}\Gamma\left(n^*-l^*+2\right)\Gamma\left(n^*+l^*+1\right)}{\left(\left(n^*-l^*-1\right)!\right)^2\Gamma(3)\Gamma\left(2+2l^*\right)}\\ &{}_3F_2\left(-n^*+l^*+1,2l^*-1,-2;-n+2l^*-1,2l^*+2;1\right)\\ &-Z^*\frac{\left(2\frac{Z^*}{n^*}\right)^{-2l^*}\Gamma\left(n^*-l^*+1\right)\Gamma\left(n^*+l^*+1\right)}{\left(\left(n^*-l^*-1\right)!\right)^2\Gamma(2)\Gamma\left(2+2l^*\right)}{}_3F_2\\ &\left(-n^*+l^*+1,2l^*,-1;-n^*+3l^*+1,2l^*+2;1\right)\end{aligned} \quad (36)$$

It is note worthy to note that the first factor $T_{nc-ni}\left(Z^*,n^*,l^*\right)$ produced with the noncommutative geometry of space, while the second term $T_3\left(Z^*,n^*,l^*\right)$ produced from the noncommutativity of phases.

4.2 The Exact Magnetic Spectrum Modifications for Modified Weakest Bound Potential for Higher Excited States

Now consider physically meaningful phenomena, it's possible to found another automatically symmetry for the production of the perturbative terms of modified weakest bound potential $H_{\text{nc-ni}}\left(\hat{r}\right)$ related to the influence of an external uniform magnetic field, it's sufficient to apply the following replacements:

$$\begin{cases}\vec{\Theta}\to\chi\vec{B}\\ \bar{\vec{\theta}}\to\bar{\sigma}\vec{B}\end{cases}\Rightarrow\left(\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\bar{\vec{\theta}}}{2}\right)\Rightarrow\left(\chi\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\sigma}}{2}\right)\vec{B}\vec{L} \quad (37)$$

Here χ and $\bar{\sigma}$ are infinitesimal real proportional's constants, and we choose the magnetic field $\vec{B}=B\vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian H_{m-ni} in (NC: 3D-RSP) as:

$$H_{m-ni}=\left(\chi\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4}-\frac{Z^*}{2r_i^3}\right)+\frac{\bar{\sigma}}{2}\right)\left(\vec{B}\vec{J}-\vec{S}\vec{B}\right) \quad (38)$$

here $\left(-\vec{S}\vec{B}\right)$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy $E_{\text{mag-ni}}$, we just replace: k_+, Θ and $\bar{\theta}$ in the eq.(34) by the following parameters: m, χ and $\bar{\sigma}$, respectively:

$$E_{\text{mag-ni}}=\left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}Bm\left\{\chi T_{nc-ni}+\frac{\bar{\sigma}}{2}T_3\right\} \quad (39)$$

We have $-l\le m\le +l$, which allow us to fixing $(2l+1)$ values for discreet number m.

5. THE EXACT MODIFIED OF n^{th} EXCITES STATES FOR MODIFIED WEAKEST BOUND POTENTIAL IN (NC: 3D- RSP)

In the light of the results of the preceding sections, let us resume the modified eigenenergies ($E_{\text{nc -uni}}$ - $E_{\text{nc -dni}}$) of a particle fermionic with spin up and spin down for modified Schrödinger equation obtained in this paper, the total modified energies are determined corresponding higher n^{th} excited states, respectively, for modified weakest bound potential in (NC: 3D-RSP), on based to original results presented on the Eqs. (34), (35) and (39), in addition to original results of energy for commutative space (9):

$$\varepsilon\Rightarrow E_{\text{nc -uni}}\left(j,Z^*,n^*,l^*,m\right)=-\frac{\left(Z^*\right)^2}{2n^{*2}}+\left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}\left\{k_+\left\{\Theta T_{nc-ni}+\frac{\bar{\theta}}{2}T_3\right\}+Bm\left\{\chi T_{nc-ni}+\frac{\bar{\sigma}}{2}T_3\right\}\right\} \quad (40)$$

$$\varepsilon\Rightarrow E_{\text{nc -dni}}\left(j,Z^*,n^*,l^*,m\right)=-\frac{\left(Z^*\right)^2}{2n^{*2}}+\left(\frac{2Z^*}{n^*}\right)^{2l^*+3}\left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}\left\{k_-\left\{\Theta T_{nc-ni}+\frac{\bar{\theta}}{2}T_3\right\}+Bm\left\{\chi T_{nc-ni}+\frac{\bar{\sigma}}{2}T_3\right\}\right\} \quad (41)$$

Thus, the original energy levels at higher excited states will be changed in the framework of extended quantum mechanics as follows:

$$\varepsilon \Rightarrow -\frac{\left(Z^*\right)^2}{2n^{*2}} + \left(\frac{2Z^*}{n^*}\right)^{2l^*+3} \left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1}$$

$$\begin{cases} \left\{k_+\left\{\Theta T_{nc-ni} + \frac{\bar{\theta}}{2}T_3\right\} + Bm\left\{\chi T_{nc-ni} + \frac{\bar{\sigma}}{2}T_3\right\}\right\} & \text{for-spin up} \\ \left\{k_-\left\{\Theta T_{nc-ni} + \frac{\bar{\theta}}{2}T_3\right\} + Bm\left\{\chi T_{nc-ni} + \frac{\bar{\sigma}}{2}T_3\right\}\right\} & \text{for-spin down} \end{cases} \quad (42)$$

It is evident to consider the quantum number m takes $(2l+1)$ values and we have also two values for $j = l \pm \frac{1}{2}$, thus every state in usually three dimensional space of energy for weakest bound potential will be $2(2l+1)$ sub-states in (NC: 3D-RSP). It's clearly, that the obtained eigenvalues of energies are real's and then the noncommutative diagonal Hamiltonian $\hat{H}_{nc-ni}$ is Hermitian$\left(\hat{H}_{nc-ni} = \left(\hat{H}_{nc-ni}\right)^+\right)$, furthermore it's possible to writing the three elements: $\left(H_{nc-ni}\right)_{11}$, $\left(\hat{H}_{nc-ni}\right)_{22}$ and $\left(\hat{H}_{nc-ni}\right)_{33}$ as follows:

$$\begin{aligned} \left(\hat{H}_{nc-ni}\right)_{11} &= -\frac{\Delta}{2} + -\frac{Z^*}{r_i} + \frac{\left[d(d+1)+2dl\right]}{r_i^2} + \\ &+ k_+\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\theta}}{2}\right) \\ &+ \left(\chi\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\sigma}}{2}\right)\left(\vec{B}\vec{J} - \vec{S}\vec{B}\right) \end{aligned} \quad (43)$$

$$\begin{aligned} \left(\hat{H}_{nc-ni}\right)_{22} &= -\frac{\Delta}{2} + -\frac{Z^*}{r_i} + \frac{\left[d(d+1)+2dl\right]}{r_i^2} + \\ &+ k_-\left(\Theta\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\theta}}{2\mu}\right) \\ &+ \left(\chi\left(\frac{\left[d(d+1)+2dl\right]}{r_i^4} - \frac{Z^*}{2r_i^3}\right) + \frac{\bar{\sigma}}{2}\right)\left(\vec{B}\vec{J} - \vec{S}\vec{B}\right) \end{aligned} \quad (44)$$

$$\left(\hat{H}_{nc-ni}\right)_{33} = -\frac{\Delta}{2} + -\frac{Z^*}{r_i} + \frac{\left[d(d+1)+2dl\right]}{r_i^2} \quad (45)$$

For, the non-weakest, we replace $k_+(k_-)$ by the coupling $S_{\overline{S}\overline{L}}$, thus we have, the modified energy $E_{\text{nc -ni}}$ as follows:

$$E_{\text{nc -ni}}\left(j, Z^*, n^*, l^*, m, S_{\overline{S}\overline{L}}\right) = -\frac{\left(Z^*\right)^2}{2n^{*2}} + \left(\frac{2Z^*}{n^*}\right)^{2l^*+3} \left[\frac{2n^*}{\left(n^*-l^*-1\right)!}\Gamma\left(n^*-l^*+1\right)\right]^{-1} \left\{S_{\overline{S}\overline{L}}\left\{\Theta T_{nc-ni} + \frac{\bar{\theta}}{2}T_3\right\} + Bm\left\{\chi T_{nc-ni} + \frac{\bar{\sigma}}{2}T_3\right\}\right\} \quad (46)$$

Thus, the total complete degeneracy of higher energy level of the modified (WBEPM) theory in noncommutative 3-dimensions spaces-phases, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\sum_{i=0}^{n-1} 2(2l+1) \equiv 2n^2 \quad (47)$$

Note that the obtained new higher energy eigenvalues ($E_{\text{nc -uni}}$ - $E_{\text{nc -dni}}$) now depend to new discrete atomic quantum numbers $\left(j, l^*\right)$ and m, in addition to effective nuclear charge Z^* and effective and effective azimuthal quantum number n^*.

6. CONCLUSION

In the present work, we reviewed the exact solutions of the Schrödinger equation with the weakest bound potential and the formalism of Bopp's shift method. Then, we have applied the Bopp's shift method to solve the modified Schrodinger equation for modified weakest bound potential in (NC: 3D-RSP), we obtained in present research paper:

The exact energy spectrum ($E_{\text{nc -uni}}$ - $E_{\text{nc -dni}}$) for higher n^{th} excited levels for the weakest bound electrons and the non-weakest bound electrons.

The modified Hamiltonian operator $\hat{H}_{nc-ni}$ for the modified weakest bound potential,

We shown that the old states are changed radically and replaced by degenerated new states, describing two new original spectrums, the first new one, produced by spin-orbital interaction while the second new spectrum produced by an external magnetic field,

We have shown that, every state in usually three dimensional space of energy for weakest bound potential will be $2(2l+1)$ sub-states in (NC: 3D-RSP).

Finally, our obtained results can find many applications to develop the (WBEPM) theory.

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