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Generalized Fokker-Planck Equation for the Modified Landau-Lifshitz Equation with Poisson White Noise

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Using the modified stochastic Landau-Lifshitz equation driven by Poisson white noise, we derive the generalized Fokker-Planck equation for the probability density function of the nanoparticle magnetic moment. In our calculations we employ the Ito interpretation of stochastic equations and take into account the fact that the magnetic moment direction can be changed by this noise instantaneously. The analysis of the stationary solution of the generalized Fokker-Planck equation shows that the distribution of the free magnetic moment in Poisson white noise is not uniform. This feature of the stationary distribution arises from using the Ito interpretation of the stochastic Landau-Lifshitz equation.

Keywords: Stochastic Landau-Lifshitz equation, Poisson white noise, Probability density function, Generalized Fokker-Planck equation.

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1. INTRODUCTION

Magnetic nanoparticles have attracted great interest during the last years of specialists studying the nanomaterials and their physical properties. Such interest is conditioned by both their wide use in many fields of science and technology and perspectives of their possible applications [1-3]. For example, magnetic nanoparticles of iron oxides are used in waste separation techniques, targeted drug delivery, hypothermic treatment of cancer cells, etc. On the other hand, investigation of the magnetic properties of nanomaterials allows to significantly broaden the field of their application and effectively use in magnetic information recording, sensors on the giant magnetoresistance effect, magnetic locks, ferrofluids, etc.

Because of external and internal fluctuations which are an integral part of real systems, dynamics of the nanoparticle magnetic moment is random. In many cases to describe the magnetic moment behavior one can use the stochastic Landau-Lifshitz equation, in which influence of fluctuations is taken into account by inclusion into effective magnetic field of an additional term with specified statistical properties. When investigating the role of thermal fluctuations, this term is usually approximated by the time-dependent random vector, whose components are independent Gaussian white noises. In this approximation, dynamics of the magnetic moment is Marcovian, and probability density of some direction of the magnetic moment satisfies the differential Fokker-Planck equation [4-6]. In particular, features of the magnetic relaxation in two-dimensional ensembles of ferromagnetic nanoparticles with magnetodipole interaction [7, 8], properties of the induced magnetization of systems of non-interacting and interacting nanoparticles in a circularly-polarized magnetic field [9-11], and dependence of the mean time between sequential reorientations of the magnetic moment on the characteristics of this field [12] are studied within the give approach.

Due to the central limiting theorem [13], approximation of the Gaussian white noise is quite justified if the corresponding random process generating this noise can be interpreted as a result of a large number of random factors, none of which is dominant. In the opposite case, non-Gaussian white noise appropriate for the description of a specific situation can be used in the modeling of random dynamics of the magnetic moment. Levi and Poisson white noises are of particular interest among the variety of such noises. Since the first noise is generated by stable Levi process, its significance results from the generalized central limiting theorem, according to which only stable distributions have a domain of attraction [14]. On the other hand, Poisson white noise representing a random sequence of delta pulses distributed by the Poisson law is a suitable model for the description of system dynamics including the dynamics of the nanoparticle magnetic moments subjected to strong but short-term impacts.

The aim of the present work is to derive the equation for the probability density of the magnetic moment of a single-domain nanoparticle in a random magnetic field, whose components have characteristics of the Poisson white noise. Since the dynamics of magnetic moment is found to be rather complex, we have used the modified Landau-Lifshitz equation for the solution of this problem.

2. MODEL AND INITIAL EQUATIONS

We consider the simplest model of a single-domain ferromagnetic nanoparticle which is characterized by the magnetic moment $\mathbf{m} = \mathbf{m}(t)$ with $m = |\mathbf{m}| = \text{const}$. In this case, random dynamics of the vector $\mathbf{m}$ can be described by the stochastic Landau-Lifshitz equation

$$\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times (\mathbf{H}_{\text{eff}} + \mathbf{h}) - \frac{\alpha\gamma}{m} \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \qquad (1)$$

where $\gamma(>0)$ is the gyromagnetic ratio; $\alpha(>0)$ is the damping parameter; sign $\times$ denotes the vector product; $\mathbf{H}_{\text{eff}} = -\partial W/\partial \mathbf{m}$ is the effective magnetic field acting on the magnetic moment; W is the nanoparticle magnetic energy, and $\mathbf{h} = \mathbf{h}(t)$ is the random magnetic field. Introducing the dimensionless time $\tau = \gamma H_0 t$ and energy $E = W/mH_0$ (H_0 is the characteristic magnetic field as which one can choose the anisotropy field), equation (1) in spherical coordinates is reduced to the system of two equations

$$\dot{\theta} = f_1 + \psi_1, \quad \dot{\varphi} = f_2 + \frac{1}{\sin\theta}\psi_2 \tag{2}$$

for polar $\theta = \theta(t)$ and azimuth $\varphi = \varphi(t)$ angles of the vector $\mathbf{m}$. Here dot above θ and φ denotes differentiation with respect to the dimensionless time,

$$\begin{aligned} f_1 &= f_1(\theta,\varphi,\tau) = -\left(\alpha\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)E, \\ f_2 &= f_2(\theta,\varphi,\tau) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta} - \frac{\alpha}{\sin\theta}\frac{\partial}{\partial\varphi}\right)E, \end{aligned} \tag{3}$$

and values $\psi_{1,2} = \psi_{1,2}(\theta,\varphi,\tau)$ are expressed through the dimensionless components $g_\kappa = h_\kappa(\tau)/H_0 (\kappa = x,y,z)$ of the random magnetic field $\mathbf{h}$ as follows:

$$\begin{aligned} \psi_1 &= -\sin\varphi\, g_x + \cos\varphi\, g_y, \\ \psi_2 &= \sin\theta\, g_z - \cos\theta\,(\cos\varphi\, g_x + \sin\varphi\, g_y). \end{aligned} \tag{4}$$

In the sequel, we will approximate components g_κ by white noises with specified statistical characteristics. Since, according to (4), these noises are multiplicative ones, the system of stochastic equations (2) should be thoroughly defined. To this end, we will re-write (2) in the differential form and use the Ito interpretation [15] (see also [5, 6]) of this system. As a result, we have

$$d\theta = f_1 d\tau + d\eta_1, \quad d\varphi = f_2 d\tau + \frac{1}{\sin\theta} d\eta_2, \tag{5}$$

where

$$\begin{aligned} d\eta_1 &= -\sin\varphi\, d\zeta_x + \cos\varphi\, d\zeta_y, \\ d\eta_2 &= \sin\theta\, d\zeta_z - \cos\theta\left(\cos\varphi\, d\zeta_x + \sin\varphi\, d\zeta_y\right), \end{aligned} \tag{6}$$

$d\zeta_\kappa = d\zeta_\kappa(\tau) = \int_\tau^{\tau+d\tau} d\tau' g_\kappa(\tau')$ with $\tau = kd\tau\ (k = 0.1, \ldots)$. As for random functions $d\zeta_\kappa(\tau)$ it is assumed that at different κ and/or k they are independent, equally distributed, and have zero mean values. This means that conditions $\langle d\zeta_\kappa(\tau)\rangle = 0$, $\langle d\zeta_\kappa(\tau) d\zeta_{\kappa'}(\tau')\rangle = 0$ ($\kappa \neq \kappa'$ and/or $\tau \neq \tau'$), $\langle[d\zeta_x(\tau)]^2\rangle = \langle[d\zeta_y(\tau)]^2\rangle = \langle[d\zeta_z(\tau)]^2\rangle$, where angular brackets denote averaging over realizations of random functions $d\zeta_\kappa(\tau)$, should hold.

Using these mean values and formulas (6), one can show that $\langle d\eta_p(\tau)\rangle = 0\ (p = 1,2)$, $\langle d\eta_p(\tau) d\eta_{p'}(\tau')\rangle = 0$ ($p \neq p'$ and/or $\tau \neq \tau'$) and $\langle[d\eta_p(\tau)]^2\rangle = \langle[d\zeta_x(\tau)]^2\rangle$. Thus, random functions $d\zeta_\kappa(\tau)$ and $d\eta_p(\tau)$ have zero mean values and equal correlation functions. In this connection it is reasonable to consider the modified system of equations (5) in which increments $d\eta_p$ are induced by white noises $\xi_p(\tau)$ with the same statistical characteristics that $g_\kappa(\tau)$ does. In other words, we represent $d\eta_p$ in the form of

$$d\eta_p = \int_\tau^{\tau+d\tau} d\tau' \xi_p(\tau') \tag{7}$$

and assume that independent noises $\xi_1(\tau)$ and $\xi_2(\tau)$ have the same properties that $g_x(\tau)$ does. For the avoidance of misunderstanding we immediately note that in the statistical meaning system of equations (5) with $d\eta_p$ from (6) is not equivalent to this system with $d\eta_p$ from (7). The reason is obvious – in the first case increments $d\eta_p$ depend on the angles θ and φ, while in the second case such dependence is absent. Nevertheless, study of the statistical properties of the magnetic moment within the modified Landau-Lifshitz equation is fully justified, since, on the one hand, influence of non-Gaussian white noises on the dynamics of $\mathbf{m}$ was not considered earlier, and, on the other hand, modified equation is simpler than the initial one.

In the present work we investigate an important from the point of view of applications case when white noise $\xi(\tau)$ (for simplification of designations we omit index p) is the Poisson noise, i.e.

$$\xi(\tau) = \sum_{i=1}^{n(\tau)} z_i \delta(\tau - \tau_i). \tag{8}$$

Here z_i are the independent random values distributed in the interval $(-\infty, +\infty)$ with some probability density $q(z)$ and which have zero mean values; $\delta(\tau)$ is the Dirac delta function; τ_i are the moments of generation of delta pulses; $n(\tau)$ is the counting Poisson process which is characterized by the probability $P(n(\tau) = n) = (\nu\tau)^n e^{-\nu\tau}/n!$ (parameter ν denotes the process intensity) that $n(\tau) \geq 0$ delta pulses in the range of $(0,\tau]$ took place. Using (7) and (8) one can show [16] that the probability density $p(\mu; d\tau) = \langle\delta(\mu - d\eta)\rangle$ that during time $d\tau$ a jump, whose value $d\eta = \int_\tau^{\tau+d\tau} d\tau' \xi(\tau')$ is equal to μ, will occur is given by the expression

$$p(\mu; d\tau) = (1 - \nu d\tau)\delta(\mu) + \nu d\tau q(\mu). \tag{9}$$

Since probability $1 - \int_{-z_0}^{z_0} dz q(z)$ of the values z with $|z| > z_0$ (z_0 is an arbitrary positive parameter) in the general case differs from zero, direction of the vector $\mathbf{m}$ in the moments of generation of delta pulses is abruptly changed. It is important to emphasize that this change can occur by multiple rotation of the magnetic moment. Indeed, since $d\eta_p \in (-\infty, \infty)$ and, consequently, $d\theta, d\varphi \in (-\infty, \infty)$, and $\theta(\tau) \in [0, \pi]$ and $\varphi(\tau) \in [0, 2\pi)$, from the condition $\mathbf{m}\big(\theta(\tau + d\tau), \varphi(\tau + d\tau)\big) = \mathbf{m}(\theta(\tau) + d\theta, \varphi(\tau) + d\varphi)$ one can obtain the following correlations:

$$\begin{aligned} \theta(\tau + d\tau) &= \theta(\tau) + d\theta - 2\pi n, \\ \varphi(\tau + d\tau) &= \varphi(\tau) + d\varphi - 2\pi m \end{aligned} \tag{10}$$

and

$$\begin{aligned} \theta(\tau + d\tau) &= -\theta(\tau) - d\theta - 2\pi n, \\ \varphi(\tau + d\tau) &= \varphi(\tau) + d\varphi - 2\pi m - \pi. \end{aligned} \tag{11}$$

Here, n and m are the integer numbers which at the specified $d\theta$ and $d\varphi$ define the number and direction of revolutions of the vector $\mathbf{m}$ in the corresponding planes.

Thus, system of equations (5) with Poisson white noise is now completely defined, and we can find the corresponding generalized Fokker-Planck equation.

3. GENERALIZED FOKKER-PLANCK EQUATION

We determine conventionally the probability density $P = P(\theta, \varphi, \tau)$ that $\theta(\tau) = \theta$ and $\varphi(\tau) = \varphi$

$$P = \langle\delta\big(\theta - \theta(\tau)\big)\delta\big(\varphi - \varphi(\tau)\big)\rangle, \tag{12}$$

where angular brackets can be interpreted as averaging over realizations of random functions $\theta(\tau)$ and $\varphi(\tau)$. Using correlations (10) and (11), probability density in the time moment $\tau + d\tau$, $\tilde{P} = P(\theta, \varphi, \tau + d\tau)$, can be represented in the form

$$\tilde{P} = \sum_{n,m}\langle[\delta(\theta - \theta(\tau) - d\theta + 2\pi n)\delta(\varphi - \varphi(\tau) - d\varphi + 2\pi m) + \delta(\theta + \theta(\tau) + d\theta + 2\pi n) \times \delta(\varphi - \varphi(\tau) - d\varphi + 2\pi m + \pi)]\rangle. \quad (13)$$

It is important to note that averaging in (13) is performed over realizations of random functions $\theta(\tau + d\tau)$ and $\varphi(\tau + d\tau)$. Taking into consideration independence of random values $d\eta_p$ from $\theta(\tau)$ and $\varphi(\tau)$, it is convenient to realize averaging in two steps. On the first step, one should carry out averaging over random values $d\eta_p$, and then average the result over realizations $\theta(\tau)$ and $\varphi(\tau)$. With taking into account equations (5) this gives

$$\tilde{P} = \sum_{n,m}\int_0^\pi d\theta' \int_0^{2\pi} d\varphi'\, P(\theta',\varphi',\tau) \iint_{-\infty}^{\infty} d\mu_1 d\mu_2 \times p(\mu_1; d\tau)p(\mu_2; d\tau)[\delta(\theta - \theta' - f_1' d\tau - \mu_1 + 2\pi n) \times \delta(\varphi - \varphi' - f_2' d\tau - \mu_2/\sin\theta' + 2\pi m) + \delta(\theta + \theta' + f_1' d\tau + \mu_1 + 2\pi n) \times \delta(\varphi - \varphi' - f_2' d\tau - \mu_2/\sin\theta' + 2\pi m + \pi)], \quad (14)$$

where $f_{1,2}' = f_{1,2}(\theta', \varphi', \tau)$.

Now, using (13) and (14), we will find the derivative $\partial P/\partial\tau = \lim_{d\tau\to 0}(\tilde{P} - P)/d\tau$. To this end, it is enough to take into consideration only linear in $d\tau$ terms in (14). Writing in this approximation

$$p(\mu_1; d\tau)p(\mu_2; d\tau) = (1 - 2\nu d\tau)\delta(\mu_1)\delta(\mu_2) + \nu d\tau \delta(\mu_1)q(\mu_2) + \nu d\tau q(\mu_1)\delta(\mu_2) \quad (15)$$

and using the delta function properties, after a number of transformations we obtain the desired generalized Fokker-Planck equation

$$\frac{\partial}{\partial\tau}P = -\frac{\partial}{\partial\theta}f_1 P - \frac{\partial}{\partial\varphi}f_2 P - 2\nu P + \nu\sin\theta \int_0^{2\pi} d\varphi' P(\theta,\varphi',\tau) \sum_m q(\sin\theta[\varphi - \varphi' + 2\pi m]) + \nu \int_0^\pi d\theta' P(\theta',\varphi,\tau) \sum_n q(\theta - \theta' + 2\pi n) + \nu \int_0^\pi d\theta' [P(\theta', \varphi + \pi, \tau)|_{\varphi<\pi} + P(\theta', \varphi - \pi, \tau)|_{\varphi>\pi}] \times \sum_n q(-\theta - \theta' + 2\pi n). \quad (16)$$

This equation, in contrast to the usual Fokker-Planck equation, is integro-differential one that is conditioned by abrupt behavior of the direction of the magnetic moment vector under the action of the Poisson noise. As usual, solution of equation (16) should be normalized, $\int_0^\pi d\theta \int_0^{2\pi} d\varphi\, P(\theta,\varphi,\tau) = 1$, and satisfy some initial condition $P(\theta,\varphi,0) = P_0(\theta,\varphi)$.

4. ANALYSIS OF THE GENERALIZED FOKKER-PLANCK EQUATION

We consider some properties of the solutions of the generalized Fokker-Planck equation (16). First of all we note that probability density $P(\theta,\varphi,\tau)$ is nonnegative function. This directly follows from the definition (12) and delta function properties. Normalization condition $\int_0^\pi d\theta \int_0^{2\pi} d\varphi\, P(\theta,\varphi,\tau) = 1$ also follows from this definition. However, to additionally verify the correctness of equation (16), it is also reasonable to show that this equation remains normalization. To this end, we will integrate both sides of equation (16) over all allowed values of angles θ and φ. Then, changing the sequence order of differentiation and integration operations and taking into account the normalization condition, it is easy to see that the left side is equal to zero. Similarly, transforming the right side with taking into consideration the fact that functions P and $f_{1,2}$ are periodic in variable φ with period 2π, it is easy to ascertain that if

$$\int_0^{2\pi} d\varphi[(f_1 P)|_{\theta=\pi} - (f_1 P)|_{\theta=0}] = 0, \quad (17)$$

then integrated equation (16) takes the form $0 = 0$. Thus, equation (16) remains normalization of P if condition (17) holds.

Now consider the simplest case of the free magnetic moment which interacts only with the Poisson noise. In this case, $f_1 = f_2 = 0$, and stationary solution of equation (16), $P_{\text{st}} = \lim_{\tau\to\infty} P(\theta,\varphi,\tau)$, can be obtained in the form $P_{\text{st}} = F(\theta)$. According to (16), function $F(\theta)$ satisfies the following equation:

$$2F(\theta) = \sin\theta\, F(\theta) \int_0^{2\pi} d\varphi' \sum_m q(\sin\theta \times [\varphi - \varphi' + 2\pi m]) + \int_0^\pi d\theta' F(\theta') \times \sum_n [q(\theta - \theta' + 2\pi n) + q(-\theta - \theta' + 2\pi n)], \quad (18)$$

which, taking into account correlation

$$\int_0^{2\pi} d\varphi' \sum_m q(\sin\theta\,[\varphi - \varphi' + 2\pi m]) = \frac{1}{\sin\theta}, \quad (19)$$

is reduced to the form

$$F(\theta) = \int_0^\pi d\theta' F(\theta') \sum_n [q(\theta - \theta' + 2\pi n) + q(-\theta - \theta' + 2\pi n)]. \quad (20)$$

Using correlation

$$\sum_n \int_0^\pi d\theta [q(\theta - \theta' + 2\pi n) + q(-\theta - \theta' + 2\pi n)] = \int_{-\infty}^{\infty} dx q(x) = 1, \quad (21)$$

it is easy to verify that integral equation (20) is compatible with normalization condition $\int_0^\pi d\theta \int_0^{2\pi} d\varphi\, F(\theta) = 1$ of the stationary probability density $F(\theta)$. It is also interesting to note that solution of equation (20) does not describe uniform distribution of the magnetic moment as one would expect based on the physical considerations. Indeed, representation $F(\theta) = \sin\theta/4\pi$ would take place in the case of uniform distribution. However, this function is not the solution of equation (20), since at $\theta = 0$ its left side is equal to zero and right one – is not. The fact that probability density $F(\theta) = \sin\theta/4\pi$, corresponding to the uniform distribution of the magnetic moment, is not the solution of stationary equation (20) follows from the use of the Ito interpretation of the initial system of stochastic equations (2). In connection with this we note that the same situation, namely, the discrepancy of stationary solutions of the Fokker-Planck equation at different interpretations of the Langevin equation with multiplicative noise, takes place also in the simplest case of the Gaussian white noise [5].

5. CONCLUSIONS

Within the modified Landau-Lifshitz equation stochastic dynamics of the nanoparticle magnetic moment induced by the Poisson white noise is considered for the first time. Using the Ito interpretation of stochastic equations and a step-wise change of the magnetic moment direction, we have derived the generalized Fokker-Planck equation for the probability density of its orientation. Analysis of this equation, whose key feature is its integral character, has shown that change in the probability density occurs with conservation of its normalization. It is also established that due to the use of the Ito interpretation of the Landau-Lifshitz equation, stationary distribution of the free magnetic moment is not uniform.

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