J. Nano- Electron. Phys.
2 (2010) No4, P. 109-116

PACS number: 05.45.Xt

SELF-SYNCHRONIZATION IN A SYSTEM OF NONLINEAR VAN DER POL OSCILLATORS

I.A. Knyaz', A.N. Vitrenko

Sumy State University,
2, Rimsky-Korsakov Str., 40007 Sumy, Ukraine
E-mail: knigorua@mail.ru

The questions of self-synchronization in a system of weakly nonlinear feedback oscillators by the example of two self-oscillating van der Pol systems are discussed. It is shown that synchronization occurs in a narrow range of values of the coupling parameter; outside of this range the dynamics of the system is autonomous. It is shown that the breaking of synchronization is connected with the change in the topology of the attractors, which are phase incoherent.

Keywords: *PHASE SYNCHRONIZATION, COUPLED OSCILLATORS, FEEDBACK, SELF-SYNCHRONIZATION, CHAOTIC DYNAMICS.*

(Received 23 November 2010, in final form 14 December 2010)

1. INTRODUCTION

Synchronization in oscillating systems of different nature has been actively studied lately [1-2]. The synchronization effect is considered in the context of biological and chemical systems [3, 4], electric circuits [5], radiophysics and information security [6], etc. For dynamic systems under synchronization one should understand the coincidence of characteristic frequencies of the interacting subsystems. Ascertainment of a constant difference between current oscillation phases of the oscillators is the stronger synchronization criterion. In this case, frequency capture automatically follows from the phase capture (this phenomenon is called the phase synchronization (PS) [7]); the inverse statement is not true.

Appearance of a certain balance between phases is the consequence of the achievement of energy balance, which occurs due to the interaction of oscillating subsystems. We note that the overwhelming majority of works devoted to the synchronization problems is based on two distinctively different types of coupling (depending on the choice of the latter, "external" or "internal" synchronization is realized). In the first case, the free leading system, which acts as the external force, governs the guided system; as a result, the capture "effect" of the external force frequency by the subsystem appears [9-11]. The second case is realized in the grids of oscillators in the presence of "diffusive" coupling between the system elements [12, 13]. The authors of [14] proposed the fundamentally different coupling scheme, which implies the presence of a special controller (smoothing filter), on whose input the phase coordinates of oscillators come; the output signal multiplied by the feedback ratio (FR) is introduced into the interacting subsystems changing their time scales. This coupling type is more natural, widely used in technical devices (telecommunication, radiolocation, etc.), and can be easily realized in many real systems (neural networks, ecological system, electric circuits, and so on) [15-18].

Theoretical analysis of the systems with feedback coupling proposed in [14] is restricted by the class of the simplest models of periodic oscillators like the Poincare systems. A class of more complex models is analyzed only within the numerical experiment, and the main attention is paid to the localization of the lower boundary of FR limits, whose excess leads to the subsystem self-synchronization. Ii is shown in the given work that amplification of the feedback coupling instead of the expected full synchronization effect (as it is suggested in [14]) can lead to its destruction that is a new result. Moreover, the authors of [14] propose a theoretical outline of the analysis of the systems with feedback coupling, which within the proposed approximations is suitable for a wider class of weakly nonlinear oscillators (with simple phase-coherent topology of the attractor). Functionality of the proposed approach is confirmed by the numerical experiment.

2. THE MODEL

The model of the dynamic system with feedback coupling was constructed as follows:

$$\begin{aligned} \frac{dx_{1,2}}{dt} &= y_{1,2}, \\ \frac{dy_{1,2}}{dt} &= -\omega_{1,2}(t)x_{1,2} + \mu_{1,2}(1 - x_{1,2}^2)y_{1,2}, \end{aligned} \tag{1}$$

where $x_{1,2}$, $y_{1,2}$ are the states of the first and the second oscillators, respectively; $\omega_{1,2}(t)$ are the frequencies; $\mu_{1,2}$ are the constants. In the case when $\omega_{1,2}(t) = \text{const}$, system (1) is a popular model of vibration and non-linear dynamics theory, i.e., the van der Pol model [19-21]. Controlling influence of $\eta(t)$ multiplied by the FR κ is introduced to each subsystem in the following way:

$$\begin{aligned} \omega_1(t) &= \omega_{(0)1} + \omega_{(0)1}\kappa\eta(t), \\ \omega_2(t) &= \omega_{(0)2} - \omega_{(0)2}\kappa\eta(t), \end{aligned} \tag{2}$$

where $\omega_{(0)1,2}$ are the natural oscillation frequencies of the oscillators 1 and 2, respectively. Evolution of $\eta(t)$ is described by the equation

$$\frac{d\eta}{dt} = -a\eta + x_1 y_2 , \tag{3}$$

where $a > 0$ is controller parameter. Scheme of the filter operation is the following: two signals x_1 and y_2 form the periodic signal, whose spectrum is represented by the low-frequency (which is determined by the difference $\Omega_1 - \Omega_2$) and the high-frequency (which is determined by the sum $\Omega_1 + \Omega_2$) components, where Ω_1, Ω_2 are the medium oscillation frequencies (typical time scales) of the first and the second oscillators, respectively. Cumulative signal is run through the filter (3), which under the condition $\omega_{(0)1} + \omega_{(0)2} > a$ suppresses the high-frequency component. After filtration signal $\eta(t)$ is added to each subsystem changing their typical time scales. As a result, certain balance between time scales, $\Omega_1 = \Omega_2$, is set.

3. THEORETICAL ANALYSIS

For diagnostics and quantitative analysis of synchronization, we introduce the corresponding characteristics, namely, the signal phase and amplitude. We have to note that there is no universal way to introduce the signal phase, which would give the correct results for any dynamic systems. For systems with simple topology of the attractor (projection of the phase trajectory on a certain plane of states (x, y) rotates all the time around the origin of coordinates (not crossing and rounding it)) instantaneous phase $\varphi(t)$ can be introduced as the angle in polar coordinate system on the (x, y)-plane [10]

$$\varphi(t) = \operatorname{arctg}\big(y(t)/x(t)\big)\,. \tag{4}$$

Use of the Poincare cross-section surface in computer simulation is more convenient way for the phase determination; in this case phase is defined as

$$\varphi(t) = 2\pi\frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \quad t_n \le t \le t_{n+1}\,, \tag{5}$$

where t_n is the time, which corresponds to the n-th intersection of the Poincare surface by the phase trajectory. For systems with simple topology of the attractor, formulas (4) and (5) give almost the same results: dynamics of the instantaneous phase for both methods will slightly differ on the time intervals lesser than the typical return period of the phase trajectory to the Poincare cross-section surface [1]. Correspondingly, the signal amplitude is determined as

$$A = \sqrt{x^2 + y^2}\,, \tag{6}$$

and the signal medium frequency Ω is calculated as the mean rate of the phase change

$$\Omega = \left\langle \frac{d}{dt}\varphi(t) \right\rangle = \lim_{T\to\infty}\frac{\varphi(T) - \varphi(0)}{T}\,. \tag{7}$$

Regime of the phase synchronization means that signal phases $\varphi_{1,2}(t)$ of the interacting systems become captured, i.e.,

$$\left|\varphi_1(t) - \varphi_2(t)\right| \le \text{const}\,. \tag{8}$$

To develop the approximate theory of phase synchronization in the system (1)-(3), we will use the polar coordinates $x_{1,2} = A_{1,2}\cos\varphi_{1,2}$, $y_{1,2} = A_{1,2}\sin\varphi_{1,2}$. Rewriting the system (1)-(3) in the variables (4), (6), we have

$$\begin{aligned}
&\cos\varphi_{1,2}\frac{dA_{1,2}}{dt} - A_{1,2}\sin\varphi_{1,2}\frac{d\varphi_{1,2}}{dt} = A_{1,2}\sin\varphi_{1,2},\\
&\sin\varphi_{1,2}\frac{dA_{1,2}}{dt} + A_{1,2}\cos\varphi_{1,2}\frac{d\varphi_{1,2}}{dt} = -\omega_{1,2}A_{1,2}\cos\varphi_{1,2} + \mu_{1,2}A_{1,2}\sin\varphi_{1,2} - \\
&\qquad\qquad - \mu_{1,2}A_{1,2}^3\cos^2\varphi_{1,2}\sin\varphi_{1,2},\\
&\frac{d\eta}{dt} = -a\eta + A_1A_2\cos\varphi_1\sin\varphi_2.
\end{aligned} \tag{9}$$

Dividing the first and the second equations of the system (9) by $\sin\varphi_{1,2}$ and $\cos\varphi_{1,2}$, respectively, and subtracting the second from the first, we obtain

$$\left(\frac{\cos\varphi_{1,2}}{\sin\varphi_{1,2}}-\frac{\sin\varphi_{1,2}}{\cos\varphi_{1,2}}\right)\frac{dA_{1,2}}{dt}-2A_{1,2}\frac{d\varphi_{1,2}}{dt}=$$
$$=A_{1,2}+\omega_{1,2}A_{1,2}-\mu_{1,2}A_{1,2}\frac{\sin\varphi_{1,2}}{\cos\varphi_{1,2}}+\mu_{1,2}A_{1,2}^3\cos\varphi_{1,2}\sin\varphi_{1,2}.$$

Using relation (2) and taking into account that at small $\mu_{1,2}$ amplitude $A_{1,2}$ changes very slowly, we obtain the evolution equation for the phase of the first and the second oscillators. In this case evolution of the phase difference will be described by the equation

$$\frac{d}{dt}(\varphi_1-\varphi_2)=\frac{1}{2}\Big[(\omega_{(0)2}-\omega_{(0)1})-\kappa\eta(t)(\omega_{(0)2}+\omega_{(0)1})+$$
$$+\mu_1\left(\frac{\sin\varphi_1}{\cos\varphi_1}-A_1^2\cos\varphi_1\sin\varphi_1\right)-\mu_2\left(\frac{\sin\varphi_2}{\cos\varphi_2}-A_2^2\cos\varphi_2\sin\varphi_2\right)\Big]. \tag{10}$$

Having introduced the "slow" phase $\theta_{1,2}$ (in accordance with relation $\varphi_{1,2} = \omega_{(0)1,2}t + \theta_{1,2}$) and divided the production $\cos\varphi_1\sin\varphi_2$ of the third equation of the system (9) into rapidly and slowly oscillating components, we average equation for $\eta(t)$ and equation (10). Taking into account that filter (3) suppresses "high" frequencies and $\langle A_1\rangle \approx \langle A_2\rangle \equiv \langle A\rangle \approx$ const, we finally have

$$\begin{aligned}&\frac{d}{dt}(\theta_1-\theta_2)=\frac{1}{2}(\omega_{(0)2}-\omega_{(0)1})-\frac{1}{2}\kappa\eta(t)(\omega_{(0)2}+\omega_{(0)1}),\\&\frac{d\eta}{dt}=-a\eta-\frac{\langle A\rangle^2}{2}\sin(\theta_1-\theta_2).\end{aligned} \tag{11}$$

Substituting system (11) as the second-order equation, we obtain

$$\frac{d^2}{dt^2}(\theta_1-\theta_2)+a\frac{d}{dt}(\theta_1-\theta_2)-\frac{a}{2}(\omega_{(0)2}-\omega_{(0)1})-$$
$$-\frac{\kappa\langle A\rangle^2}{4}(\omega_{(0)2}+\omega_{(0)1})\sin(\theta_1-\theta_2)=0. \tag{12}$$

As follows, fulfillment of the equality

$$\sin(\theta_1-\theta_2)=-\frac{2a(\omega_{(0)2}-\omega_{(0)1})}{\kappa\langle A\rangle^2(\omega_{(0)2}+\omega_{(0)1})} \tag{13}$$

is the condition of phase capture that is possible if

$$\kappa\geq\frac{2a\left|\omega_{(0)2}-\omega_{(0)1}\right|}{\langle A\rangle^2(\omega_{(0)2}+\omega_{(0)1})}. \tag{14}$$

Thus, the threshold value of the FR κ_p, whose excess leads to the regime of phase synchronization, is $\kappa_p = 2a|\omega_{(0)1} - \omega_{(0)2}|/\langle A\rangle^2(\omega_{(0)1} + \omega_{(0)2})$.

In Fig. 1 we present the synchronization region of subsystems in the plane of FR parameters and difference of natural frequencies of subsystems 1 and 2. At small $\mu_{1,2}$ phase trajectory of the van der Pol oscillator represents a circle, whose radius can be easily estimated numerically. At the chosen parameters the average oscillation amplitude is $\langle A_1 \rangle = 2$. Solid curve corresponds to the boundary of the synchronization regime in accordance with the performed theoretical analysis. Presence of the synchronous regime was determined by fulfillment of the condition of the phase capture (14). As seen from Fig. 1, in the case of small frequency mismatch, synchronous regime is realized at sufficiently small values of the FR. With the linear increase in the mismatch, phase synchronization is possible under the condition of logarithmic increase in the FR.

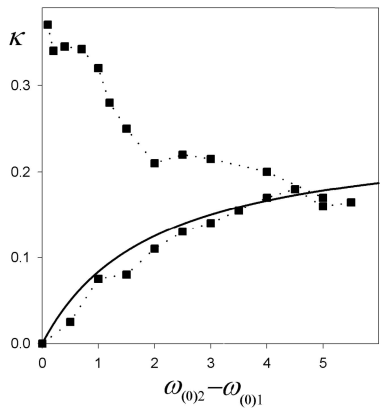

Fig. 1 *– Dependence of the critical value of the FR on the difference of natural frequencies of the interacting subsystems $\omega_{(0)2} - \omega_{(0)1}$ at $a = 0{,}5$. Region of phase synchronization calculated within the theoretical analysis is above the solid curve. Synchronization region calculated based on the numerical experiment is between the curves marked by the squares*

4. NUMERICAL ANALYSIS

Computer simulation of the synchronization processes in the represented system was performed based on the direct numerical solution of the equations (1)-(3) by the Euler-Cromer method with the integration step $h = 0{,}001$. Initial conditions were chosen randomly, transient process of the length $T = 200$ was omitted. Controlling influence was switched on in the moment $t = 200$ after trajectory running on the limit cycle.

Critical values of the FR, at which self-synchronization is possible for different combinations of the natural frequencies of subsystems, were calculated within the numerical analysis. Generalized results of the performed experiments were superimposed on the results of the theoretical calculations (see Fig. 1). As seen from the figure, calculation results of the lower boundary of the interval of FR critical values practically coincide that confirms the correctness of the used approximations and proposed scheme in whole. As computer analysis showed, further increase in the FR leads to the reduction of the transient process time during transition to the synchronous behavior after switching on of the controlling influence. At large values of the FR, the

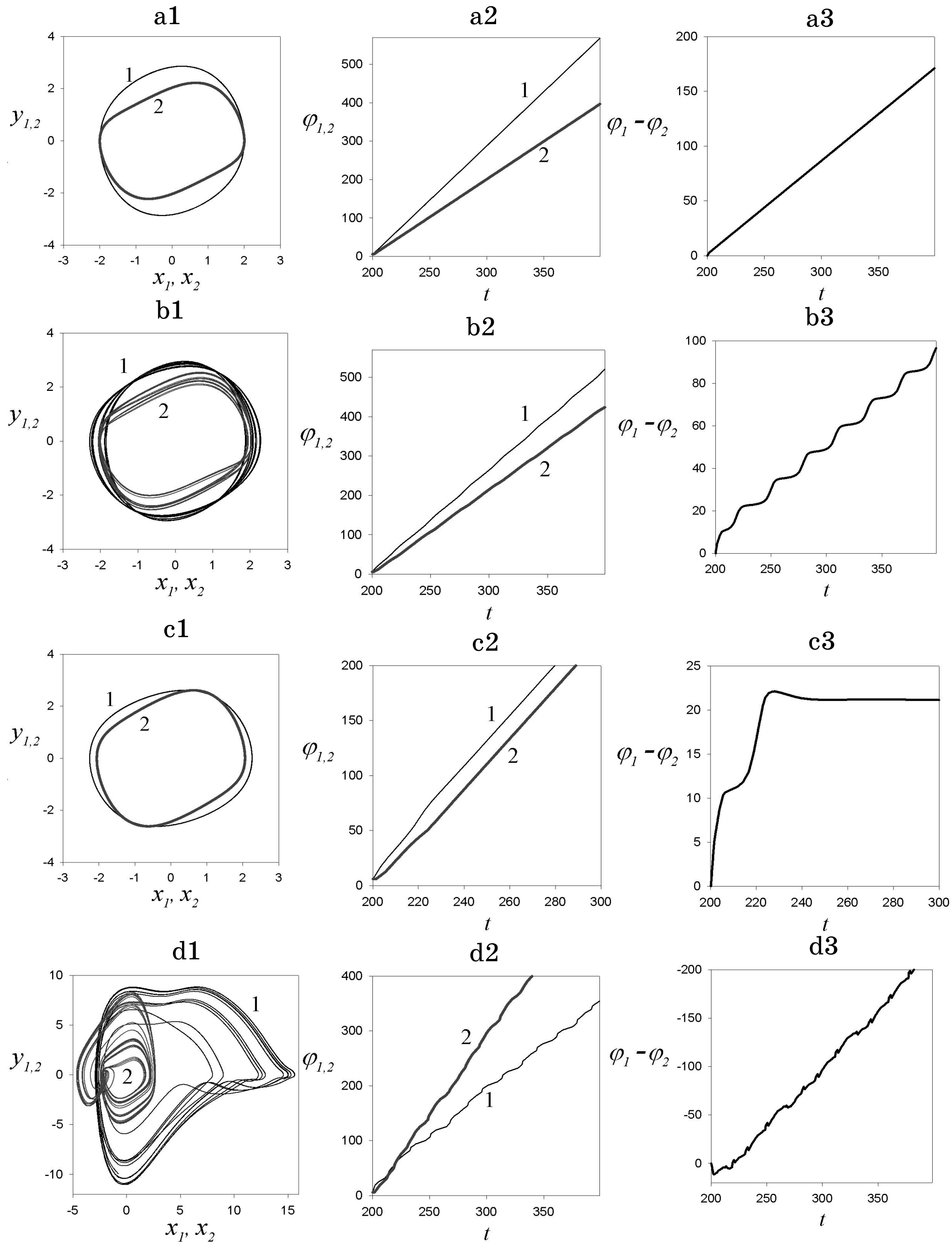

Fig. 2 *– Phase portraits of the oscillators (a1-d1), time evolution of the phases (a2-d2), and dynamics of their difference (a3-d3) on the interval* $t \in [200, 400]$ *at different values of the FR. Pictures a1-a3 correspond to the case* $\kappa = 0$ *(non-interacting oscillators); b1-b3 –* $\kappa = 0{,}06$*; c1-c3 –* $\kappa = 0{,}075$*; d1-d3 –* $\kappa = 0{,}32$*. Values of other parameters:* $\omega_{(0)1} = 2$, $\omega_{(0)2} = 1$, $\mu_1 = 0{,}2$, $\mu_2 = 0{,}5$, $a = 0{,}5$*. Curves, which correspond to the first and the second oscillators, are marked by the numerals 1 and 2, respectively*

unexpected destruction effect of synchronous dynamics and transition to chaos is observed. To identify the reasons of synchronization destruction, we have studied the dynamics of both oscillators at different FR in detail.

In Fig. 2 we represent the phase portraits of the interacting subsystems; evolution of the phases, and dynamics of their difference at different FR. Due to the absence of interaction between oscillators (Fig. 2a1-2a3), a limit cycle is the sole attractor on the phase plane. When switching on the controlling influence, phase trajectories are "blurred" in space: attractor of each of the interacting subsystem becomes chaotic (transition to the chaotic dynamics is determined by the Lyapunov exponents). With the increase in the FR up to 0,0075, phase synchronization occurs in the system that is expressed in the attainment of the steady-state level by the phase difference curve (see Fig. 2c3). Obtained critical value of the FR agrees well with the results of the theoretical analysis: in accordance with expression (14) the threshold value of the FR at the given parameters of subsystems is about $\kappa_p \approx 0{,}083$. In the range of $\kappa \in [0{,}075; 0{,}3]$ oscillations of the subsystems remain synchronized. With the increase in κ ($\kappa > 0{,}3$), instead of the expected regime of full synchronization, the inverse effect is observed: synchronization is destroyed (see Fig. 2d2-2d3), attractors of the subsystems (as it is shown in Fig. 2d1) become phase incoherent (topology becomes more complex, phase trajectory cuts the x-axis on the left of the origin of coordinates). Thus, in this case destruction of the regime of full synchronization is equivalent to the destruction of the attractor phase coherence. Since approximations of the foregoing theoretical analysis concern only the case of the presence of the phase coherent attractor, condition (14) does not take into account change in the attractor topology and, correspondingly, presence of the upper critical point in the range of FR, whose excess leads to the synchronization destruction.

5. CONCLUSIONS

In the present work, the features of realization of the self-synchronization regime in a system of similar non-linear feedback oscillators are investigated within the theoretical analysis and computer experiment. It is shown that with the increase in the feedback ratio in the narrow domain of the given parameter, the reverse transition similar to the reverse phase transitions in physical systems [22] takes place: gradual increase in the coupling parameter leads, at first, to the appearance of phase synchronization, and then to its destruction. The process of synchronization destruction is connected with the change in the topology of the subsystem attractors, which become phase incoherent.

REFERENCES

1. A.S. Pikovsky, M.G. Rosenblum, J. Kurths, *Synchronization – A Universal Concept in Nonlinear Sciences* (Cambridge: Cambridge University Press: 2001).
2. V.S. Anshchenko, V. Astakhov, A. Neiman, T. Vadivasova, L. Schimansky-Geier, *Nonlinear Dynamics of Chaotic and Sochastic Systems. Tutorial and Modern Developments* (Heidelberg: Springer-Verlag: 2001).
3. I. Schreiber, M. Marek, *Physica D* **5**, 258 (1982).
4. S.K. Han, C. Kurrer, Y. Kuramoto, *Phys. Rev. Lett.* **75**, 3190 (1995).
5. J.F. Heagy, T.L. Caroll, L.M. Pecora, *Phys. Rev. E* **50**, 1874 (1994).

6. A.S. Dmitriev, A.I. Panas, *Dinamicheskiy khaos: novye nositeli informatsii dlya system svyazi* (M.: Fizmatlit: 2002).
7. M.G. Rosemblum, A.S. Pikovsky, J. Kurths, *Phys. Rev. Lett.* **76**, 1804 (1996).
8. V.S. Afraimovich, N.N. Verichev, M.I. Rabinovich, *Izvestiya VUZov: Radiofizika* **29**, 1050 (1986).
9. L.M. Pecora, T.L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990).
10. M.G. Rosemblum, A.S. Pikovsky, J. Kurths, *Phys. Rev. Lett.* **78**, 4193 (1997).
11. L. Kocarev, U. Parlitz, *Phys. Rev. Lett.* **76**, 1816 (1996).
12. V.M. Malafeev, M.S. Polyakova, Yu.M. Romanovsky, *Izvestiya VUZov: Radiofizika* **13**, 936 (1970).
13. D.K. Mynbaev, M.I. Shilenkov, *Radiotekhnika i elektronika* No2, 361 (1981).
14. V.N. Belykh, G.V. Osipov, N. Kucklander, B. Blasius, J. Kurths, *Physica D* **200**, 81 (2004).
15. M. Rabinovich, A. Volkovskii, P. Lecanda, R. Huerta, H.D.I. Abarbanel, G. Laurent, *Phys. Rev. Lett.* **87**, 068102 (2001).
16. J.D. Murray, *Mathematical Biology* (Berlin: Springer: 1989).
17. Y. Braiman, T.A.B. Kennedy, K. Wiesenfeld, A. Khibnik, *Phys. Rev. A* **52**, 1500 (1995).
18. W.C. Lindsey, *Synchronization Systems in Communication and Control* (Prentice-Hall, Englewood Cliffs, NJ: 1972).
19. B. van der Pol, *Phil. Mag.* **3**, 65 (1927).
20. A.A. Andronov, A.A. Witt, Yu.E. Khaikin, *Teoriya kolebaniy* (M.: Gostekhizdat: 1937).
21. N. Minorsky, *Nonlinear Oscillations* (Princeton, NJ: Van Nostrand: 1962).
22. D.O. Kharchenko, I.A. Knyaz, *J. Phys. Studies* **8**, 299 (2004).